Qualitative properties of nonlocal discrete operators

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Abstract

In this article, we prove new convexity results for nonlocal discrete operators across an entire region that covers sequential orders in two parameters. We review and extend current studies on the properties of positivity, monotonicity and convexity, explore borderline cases, and provide new insights on such properties by means of original examples evidencing the sharpness of the results. Our method is based on the novel principle of transference.

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1. Introduction: Background and main results.

The study of nonlocal discrete equations has experienced a rapid increase in the last few years opening new and fresh lines of research. This theory emerged as a discrete counterpart of the more classical theory of fractional differential equations. As is by now well known, at the basis of fractional differential equations there are two basic nonlocal operators of interest: The nonlocal extension of *time* differential operators, e.g. the Riemann-Liouville fractional differential operator, and the nonlocal version of differential operators in space, e.g. the fractional Laplacian. This paper is concerned with the discrete version of the first one, whose more studied definition is the following (see Gray and Zhang [18] or Atici and Eloe [2, 3, 4, 5], where the first definitions of the nabla and delta operators were introduced)

$$\big(\Delta_a^{\nu} f\big)(t) := \frac{1}{\Gamma(-\nu)} \sum_{s=a}^{t+\nu} (t-s-1)^{\underline{-\nu-1}} f(s),\ t \in \mathbb{N}_{a+N-\nu}, \tag{1.1}$$

where $N \in \mathbb{N}$ is the unique integer satisfying $N-1 < \nu < N$, and the map $t \mapsto t^{\underline{\nu}}$ is defined by $t^{\underline{\nu}} := \dfrac{\Gamma(t+1)}{\Gamma(t+1-\nu)}$. Note that here and in the sequel, we utilize the standard notation $\mathbb{N}_r := \{r, r+1, ...\}$ for

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$r \in \mathbb{R}$.

Time-discrete operators of fractional order appear in several areas of interest. In numerical analysis, as time-stepping schemes of approximation for fractional evolution equations [33, 37], in the study of existence, uniqueness and qualitative properties of fractional difference equations [1, 6, 11, 12, 17, 29, 41, 42], and in the analysis of mixed partial difference-differential equations by means of operator theoretical methods [30, 34, 35, 36, 38], among others.

A well-known geometrical fact in the difference calculus is the following characterization of monotonicity: $(\Delta_a f)(t) \geq 0$ if and only if f is increasing on $\mathbb{N}_a$. The question of whether such a monotonicity result holds in the discrete fractional setting, i.e. *the monotonicity conjecture,* was posed in 2014 by Dahal and Goodrich [13, 14]. It turns out that the answer to this conjecture is not obvious and rather complicated due to the *nonlocal* character of the fractional difference operator (1.1). In fact, it was proved that if $1 < \nu < 2$ and $\big(\Delta_a^\nu f\big)(t) \geq 0$ for each $t \in \mathbb{N}_{a+2-\nu}$ one does not need to have f increasing [16, Example 2.4]. In particular, it is necessary that some additional condition be imposed [16].

On the other hand, connections between (1.1) and the convexity of the map f was first investigated by Goodrich [20], proving that under certain hypotheses the positivity of $(\Delta^\nu f)(t)$, for $2 < \nu < 3$, implies the convexity of f, thereby associating some geometrical meaning to the fractional difference operator of order $\nu > 2$.

In 2017, Dahal and Goodrich [15] consider monotonicity-type results for sequences f satisfying the sequential fractional difference inequality $\Delta^\nu_{1+a-\mu}\Delta^\mu_a f(t) \geq 0$ for $t \in \mathbb{N}_{2+a-\mu-\nu}$, where $0 < \mu < 1, 0 < \nu < 1$, and $1 < \mu + \nu < 2$. A first motivation is based on the study of discrete sequential fractional boundary problems, initiated by Goodrich [19]. See also Sitthiwirattham [40]. A second interest on this problem is due to that discrete fractional operators are, in general, noncommutative [31]. In particular, this renders reduction of the order of fractional difference equations impossible. An interesting aspect of the sequential case is that the type of result obtained depend on the choice of ν and μ [24, 26] and therefore exhibits a complexity that appears to be absent in the non-sequential case.

In particular, and after the intense work of several authors [10, 13, 21, 32], the best answer to the monotonicity conjecture posed by Dahal and Goodrich was proved in [26, Theorem 6.3]. In such paper, it was also analyzed the case of compositions of discrete fractional operators, establishing many new results for all types of discrete fractional differences, and improving existing results in the literature.

However, even though the results in the recent reference [26] are a significative advance, they are not sufficient to answer the problem of fully understanding the properties of positivity, monotonicity and convexity of fractional difference operators as a whole. In particular, the results on convexity were not enough because they did not account that even properties of positivity and monotonicity could be reached according to the author's given hypothesis. Also, the sufficiency and sharpness of hypothesis, and how these hypotheses could eventually be connected in borderline cases was not established. These open problems are our main motivation for this work.

Our main objective in this article is to completely solve the aforementioned problems, improving on the one hand the existing results, especially with regard to convexity, and provided, among others, in the reference [26] and, on the other hand, establishing a complete state of the art for the properties of positivity, monotonicity and convexity, even providing a schematic representation. We will also provide original examples that will demonstrate the sharpness of the hypothesis in a variety of cases. Finally, we will give a comprehensive answer to an open problem raised by Dahal and Goodrich in 2014 [16, Observation 2.9] regarding the inverse of the monotonicity conjecture.

Consequently, our main contributions in this article are the following:

(i) To show a geometrical interpretation for the concepts of ν-increasing and ν-convex sequence (Section 4), which seems to be completely new in the existing literature;

(ii) To provide new formulas of higher order differences Δ^l, for $l \in \mathbb{N}$ and of the $\alpha - th$ fractional difference operator Δ^α, for $\alpha > 0$ (Section 3). These results broadly generalize [26, Proposition 2.9] and [26, Theorems 5.7, 6.11, 6.16, 6.21, 7.8];

(iii) To provide original examples showing the optimality of the necessary conditions stated in our results on the positivity property (Examples 4.7, 4.14 and 4.19);

(iv) To improve the existing results about convexity (Theorem 5.1) and to provide a novel example that show optimality on the given necessary conditions (Example 5.4).

(v) To improve the existing results about sequential convexity (Theorem 5.6) generalizing [26, Theorems 7.9, 7.11, 7.13, 7.15, 7.17], and to providing original examples that show sharpness of the result (Examples 5.11, 5.12 and 5.13).

(vi) To transfer all the above mentioned advancements on understanding the Δ^α operator to the operator Δ_a^ν providing a complete picture of the geometry of two sequential operators (Figure 1).

1.1. Background

A very important point regarding (1.1) is that the operator acts on *different* spaces of sequences. This causes serious difficulties when analyzing fractional differences. However, recently, it was proved in [26] that a *transference principle* can be used to overcome such difficulty, using an alternative definition to (1.1) - via convolution - by setting

$$\big(\Delta^\alpha f\big)(n) := \Delta^N\big(k^{N-\alpha} * f\big)(n) := \Delta^N \sum_{j=0}^{n} k^{N-\alpha}(n-j)f(j), \quad n \in \mathbb{N}_0, \quad N-1 < \alpha < N, \quad N \in \mathbb{N}, \tag{1.2}$$

where the kernel $n \mapsto k^\alpha(n)$ is defined by

$$k^\alpha(n) := \frac{\Gamma(\alpha+n)}{\Gamma(\alpha)\Gamma(n+1)}, \quad n \in \mathbb{N}_0.$$

It turns out that (1.2) behaves extremely nicely allowing us to deduce results about the operator Δ^α in a clear and transparent way and then readily deduce the corresponding property for the operator Δ_a^ν. This notion of transference has been developed and used in [26] to understand the connections between the sign of $\big(\Delta^\alpha f\big)(t)$ and either the positivity, monotonicity, or convexity of f. In such reference, many new results were proved, improving most - if not all - known existing results in the literature. In particular, the best answer to the monotonicity conjecture was proved. We revisit this result in Theorem 1.4, below. We remark that positivity and monotonicity results for triple sequential fractional differences via convolution started to be studied very recently [27].

In the case of positivity, one of the main novelties of the case $0 < \nu < 1$ is that the condition that f is increasing must be replaced by the concept of ν*-increasing*, introduced by Atici and Uyanik [7, Definition 2.3], that reads as follows:

Definition 1.1. *[7, Definitions 2.3 and 2.4] Let $\nu \geq 0$ and $a \in \mathbb{R}$ be given. We say that a sequence $f : \mathbb{N}_a \to \mathbb{R}$ is $\nu-$increasing (respectively, ν-decreasing) if*

$$f(t+1) \geq \nu f(t) \tag{1.3}$$

(respectively, $f(t+1) \leq \nu f(t)$) for all $t \in \mathbb{N}_a$.

Note that for $\nu = 0$ we get the definition of positive sequence, while for $\nu = 1$ we retrieve the concept of increasing sequence.

It turns out that the concept of ν-increasing sequence, have a counterpart: ν-convex sequence, introduced in [26, Definition 6.1].

Definition 1.2. *Let $\nu \geq 1$. We say that a sequence $f : \mathbb{N}_a \to \mathbb{R}$ is ν-convex if*

$$f(t+2) - \nu f(t+1) + (\nu-1)f(t) \geq 0, \quad t \in \mathbb{N}_a. \tag{1.4}$$

Note that when $\nu = 2$ we recover the notion of convexity and when $\nu = 1$ the concept of increasing sequence in the set $\mathbb{N}_{a+1}$. Furthermore, if we assume that $f(a+1) \geq f(a)$ then the concepts of 1-increasing and 1-convex also coincide in the set $\mathbb{N}_a$.

As we will see in this paper, the notions of ν-monotonicity (increasing) and ν-convexity are extremely important to understand the geometry of the fractional difference operator (1.1) by means of the simultaneous use of the transference principle and the difference operator (1.2). As previously intimated, one of the main properties is their continuous transition from positivity (0-increasing) to monotonicity (1-increasing) and, finally, to convexity (2-convex).

1.2. Main results

We collect in this subsection the main results needed to globally understand the geometrical properties of positivity, monotonicity and convexity of (1.1) that appear in the interval $0 \leq \nu < 3$. We start with the following theorem, proved in [26, Corollary 5.6] for the open interval $0 < \alpha < 1$. We extend slightly this result to include the borders or boundary of the interval (or region) where such geometrical property holds.

Theorem 1.3. *[26, Corollary 5.6] Let $0 \leq \nu \leq 1$, $a \in \mathbb{R}$ and $v \in s(\mathbb{N}_a;\mathbb{R})$ be given a sequence. Suppose that*

(i) $\Delta_a^{\nu} v(t) \geq 0,$ *for all* $t \in \mathbb{N}_{a+1-\nu}$*;*
(ii) $v(a) \geq 0.$

Then v is positive and ν-increasing on $\mathbb{N}_a$.

Analogously, we give a slight improvement to the borders of the following theorem.

Theorem 1.4. *[26, Corollary 6.9] Let $1 \leq \nu \leq 2$, $a \in \mathbb{R}$ and $v \in s(\mathbb{N}_a;\mathbb{R})$ be given a sequence. Suppose that*

(i) $\Delta_a^{\nu} v(t) \geq 0,$ *for all* $t \in \mathbb{N}_{a+2-\nu}$*;*
(ii) $v(a+1) \geq \nu v(a)$;
(iii) $v(a) \geq 0.$

Then v is positive, increasing and ν-convex on $\mathbb{N}_a$.

Theorems 1.3 and 1.4 allow us to give an answer to the open problem posed by Dahal and Goodrich in 2014 [15, Remark 2.9] as follows: We look at the same problem but when $0 < \nu \leq 1$. In such case, we realize that if $f(a) \geq 0$ and f is increasing (and hence positive) then $\big(\Delta_a^{\nu} f\big)(t) \geq 0$ for each $t \in \mathbb{N}_{a+1-\nu}$. See Remark 4.5, below, for a proof of this claim. Note that this result was originally discovered by Atici and Uyanik [7]. In the case $1 < \nu < 2$ the key observation is that the concept of f increasing continuously change to be the concept of ν-increasing (since a sequence is increasing if and only if it is 1-increasing). Then, the conclusion in case $1 \leq \nu \leq 2$ reads: If $f(a) \geq 0$ and f is ν-increasing then $\big(\Delta_a^{\nu} f\big)(t) \geq 0$ for each $t \in \mathbb{N}_{a+1-\nu}$. See Remark 4.12 below for a proof. We then may conclude that because the two before mentioned results coincide as the parameter ν varies continuously from 0 to 2 they, together, provide a plausible answer to the open posed problem.

The previous analysis also serves to justify, in passing, why we are interested in the present work in the study of border cases when the parameters vary.

In addition, a geometrical interpretation for the concepts of ν-increasing and ν-convex sequence is given in Section 4, which seems to be completely new in the existing literature. See also Figures 2 and 3.

The next theorem is an extension of [26, Theorem 7.1] combined with the transference principle, that ensure the properties of positivity and monotonicity in the region $2 \leq \nu < 3$. We observe that the last two properties were not included in [26] and therefore are new. It is important to observe that this result allows us to conclude that the geometrical properties of positivity, monotonicity and convexity for a given sequence u have a continuous transition as α increase from 0 to 3. We give also a different proof than [26] to show the convexity, using this time the properties of the operator Δ^{α} established in Section 4, below.

Theorem 1.5. *Let $2 \leq \nu < 3$, $a \in \mathbb{R}$ and $v \in s(\mathbb{N}_a;\mathbb{R})$ be given. And assume that*

(i) $\Delta_a^\nu v(t) \geq 0$, *for all* $t \in \mathbb{N}_{a+3-\nu}$;

(ii) $v(a+2) \geq \nu v(a+1) - \frac{\nu(\nu-1)}{2} v(a)$;

(iii) $v(a+1) \geq \nu v(a)$;

(iv) $v(a) \geq 0$.

Then v is positive, increasing and convex on $\mathbb{N}_a$.

For the composition of two operators $\Delta^\nu \circ \Delta^\mu$ a complete picture of the whole geometric behavior is shown in Figure 1, next, and explained in the following results.

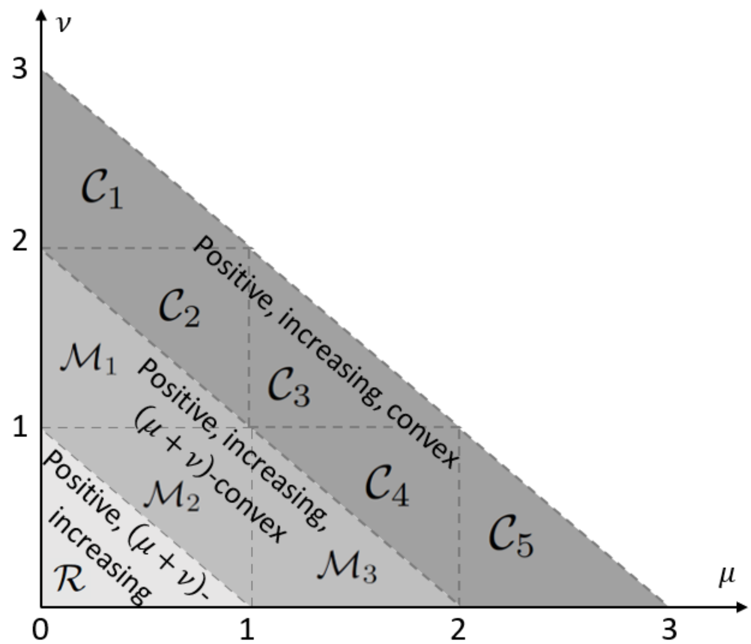

Figure 1: Geometry of the sequential operator $\Delta^\nu \circ \Delta^\mu$

The first result is a special case of [26, Corollary 5.10] combined with the transference principle, which is enough for our purposes.

Theorem 1.6. *Let $a \in \mathbb{R}$ and $v \in s(\mathbb{N}_a;\mathbb{R})$ be given. Suppose that $(\mu,\nu) \in \mathcal{R}$ and*

(i) $\left(\Delta_{1+a-\mu}^\nu \circ \Delta_a^\mu v\right)(t) \geq 0$, *for all* $t \in \mathbb{N}_{a+2-\mu-\nu}$;

(ii) $v(a+1) \geq 0$;

(iii) $v(a) = 0$.

Then v is positive and $(\mu+\nu)$-increasing on $\mathbb{N}_a$.

In the next theorem, the result in $\mathcal{M}_1$ refines [26, Corollary 6.24]. The corresponding result to $\mathcal{M}_2$ is an improvement of [26, Corollary 6.14] with an extra assumption that was not previously considered. Finally, the result in $\mathcal{M}_3$ is a substantial improvement of [26, Corollary 6.19] and proved below (Theorem 4.16). We note that a careful study of only monotonicity in the sectors $\mathcal{M}_1, \mathcal{M}_3$ and $\mathcal{M}_2$ was carry out by Goodrich (see [25] and [24], respectively). We remark that further advances in the sector $\mathcal{M}_2$ via homotopy methods have been recently appeared [28].

Theorem 1.7. *Let $a \in \mathbb{R}$ and $v \in s(\mathbb{N}_a;\mathbb{R})$ be given. Suppose that,*

(i)

$$\begin{cases} (\Delta^{\nu}_{a+1-\mu} \circ \Delta^{\mu}_a v)(t) \geq 0 & \text{for all } t \in \mathbb{N}_{a+3-\mu-\nu} & \text{if } (\mu,\nu) \in \mathcal{M}_1, \\ (\Delta^{\nu}_{a+1-\mu} \circ \Delta^{\mu}_a v)(t) \geq 0 & \text{for all } t \in \mathbb{N}_{a+2-\mu-\nu} & \text{if } (\mu,\nu) \in \mathcal{M}_2, \\ (\Delta^{\nu}_{a+2-\mu} \circ \Delta^{\mu}_a v)(t) \geq 0 & \text{for all } t \in \mathbb{N}_{a+3-\mu-\nu} & \text{if } (\mu,\nu) \in \mathcal{M}_3; \end{cases}$$

(ii)

$$\begin{cases} v(a+2) \geq (\mu+\nu)v(a+1) - \frac{1}{2}(\mu+\nu)(\mu+\nu-1)v(a); & \\ v(a+1) \geq (\mu+\nu)v(a); \quad v(a) \geq 0 & \text{if } (\mu,\nu) \in \mathcal{M}_1, \\ & \\ v(a+2) \geq (\mu+\nu)v(a+1); \quad v(a+1) \geq 0; \quad v(a) = 0 & \text{if } (\mu,\nu) \in \mathcal{M}_2, \\ & \\ v(a+2) \geq 0; \quad v(a+1) = v(a) = 0 & \text{if } (\mu,\nu) \in \mathcal{M}_3. \end{cases}$$

Then v is positive, increasing and $(\mu+\nu)$-convex on $\mathbb{N}_a$.

Our final theorem provides our new results and insights on convexity, which seems to be the best possible. The result in the region $\mathcal{C}_1$ corresponds to [26, Theorem 7.9] after application of the transference principle and adding one missing hypothesis. The result in $\mathcal{C}_2$ corresponds to a substantial improvement of [26, Theorem 7.11] where not only one hypothesis was missing but also the conclusions on positivity and monotonicity. The result in $\mathcal{C}_3$ is an extension of [26, Theorem 7.13] where, even after application of the transference principle, both an additional hypothesis and the conclusions on positivity and monotonicity were absent. For a proof, see Theorem 5.6 and Section 6, below. The result in $\mathcal{C}_4$ is a major improvement of [26, Theorem 7.15]. See Theorem 5.6 and Section 6, below. Finally, the conclusion about the sector $\mathcal{C}_5$ widely improves [26, Theorem 7.17]. See Theorem 5.6 and Section 6. We notice that our next theorem also improves those in the reference [22] for the sectors $\mathcal{C}_3, \mathcal{C}_2$ and $\mathcal{C}_4$.

Theorem 1.8. *Let $a \in \mathbb{R}$ and $v \in s(\mathbb{N}_a;\mathbb{R})$ be given. Suppose that,*

(i)

$$\begin{cases} (\Delta^{\nu}_{a+1-\mu} \circ \Delta^{\mu}_a v)(t) \geq 0 & \text{for all } t \in \mathbb{N}_{a+4-\mu-\nu} & \text{if } (\mu,\nu) \in \mathcal{C}_1, \\ (\Delta^{\nu}_{a+1-\mu} \circ \Delta^{\mu}_a v)(t) \geq 0 & \text{for all } t \in \mathbb{N}_{a+3-\mu-\nu} & \text{if } (\mu,\nu) \in \mathcal{C}_2, \\ (\Delta^{\nu}_{a+2-\mu} \circ \Delta^{\mu}_a v)(t) \geq 0 & \text{for all } t \in \mathbb{N}_{a+4-\mu-\nu} & \text{if } (\mu,\nu) \in \mathcal{C}_3, \\ (\Delta^{\nu}_{a+2-\mu} \circ \Delta^{\mu}_a v)(t) \geq 0 & \text{for all } t \in \mathbb{N}_{a+3-\mu-\nu} & \text{if } (\mu,\nu) \in \mathcal{C}_4, \\ (\Delta^{\nu}_{a+3-\mu} \circ \Delta^{\mu}_a v)(t) \geq 0 & \text{for all } t \in \mathbb{N}_{a+4-\mu-\nu} & \text{if } (\mu,\nu) \in \mathcal{C}_5; \end{cases}$$

(ii)

$$\begin{cases} v(a) = 0; \quad v(a+1) \geq 0; \quad v(a+2) \geq (\mu+\nu)v(a+1) & \\ v(a+3) \geq (\mu+\nu)v(a+2) - \frac{1}{2}(\mu+\nu)(\mu+\nu-1)v(a+1) & \text{if } (\mu,\nu) \in \mathcal{C}_1, \\ & \\ v(a) \geq 0; \quad v(a+1) \geq (\mu+\nu)v(a); \quad v(a+2) \geq (\mu+\nu)v(a+1) & \\ -\frac{1}{2}(\mu+\nu)(\mu+\nu-1)v(a); & \\ v(a+3) \geq (\mu+\nu)v(a+2) - \frac{1}{2}(\mu+\nu)(\mu+\nu-1)v(a+1) & \\ +\frac{1}{6}(\mu+\nu)(\mu+\nu-1)(\mu+\nu-2)v(a) & \text{if } (\mu,\nu) \in \mathcal{C}_2, \\ & \\ v(a) \geq 0; \quad v(a+1) \geq (\mu+\nu)v(a); \quad v(a+2) \geq (\mu+\nu)v(a+1) & \\ -\frac{1}{2}(\mu+\nu)(\mu+\nu-1)v(a); & \\ v(a+3) \geq (\mu+\nu)v(a+2) - \frac{1}{2}(\mu+\nu)(\mu+\nu-1)v(a+1) & \\ +\frac{1}{6}(\mu+\nu)(\mu+\nu-1)(\mu+\nu-2)v(a) & \text{if } (\mu,\nu) \in \mathcal{C}_3, \\ & \\ v(a) = v(a+1) = 0; \quad v(a+2) \geq 0 \quad v(a+3) \geq (\mu+\nu)v(a+2) & \text{if } (\mu,\nu) \in \mathcal{C}_4, \\ & \\ v(a) = v(a+1) = v(a+2) = 0; \quad v(a+3) \geq 0 & \text{if } (\mu,\nu) \in \mathcal{C}_5. \end{cases}$$

Then v is positive, monotone increasing and convex on $\mathbb{N}_a$.

2. Preliminaries.

In this section, we provide the necessary preliminaries on fractional differences that will be used throughout the paper. In what follows, we denote $\mathbb{N}_a := \{a, a+1, a+2, \dots\}$, for some $a \in \mathbb{R}$, and $\mathbb{N} \equiv \mathbb{N}_1$ as usual. We denote by $s(\mathbb{N}_a;\mathbb{R})$ the vectorial space that consists of all sequences $f : \mathbb{N}_a \to \mathbb{R}$. Recall that given a sequence $f \in s(\mathbb{N}_a;\mathbb{R})$ the first-order forward (or delta) difference of f at $t \in \mathbb{N}_a$, denoted $\big(\Delta_a f\big)(t)$, is defined by

$$\big(\Delta_a f\big)(t) := f(t+1) - f(t).$$

Then one may define iteratively the higher order differences Δ_a^n, for $n \in \mathbb{N}_1$, by writing

$$\big(\Delta_a^n f\big)(t) := \big(\Delta_a \circ \Delta_a^{n-1} f\big)(t).$$

We also denote $\Delta_a^0 \equiv I_a$, where $I_a : s(\mathbb{N}_a;\mathbb{R}) \to s(\mathbb{N}_a;\mathbb{R})$ is the identity operator, $\Delta_a^1 \equiv \Delta_a$, and $\Delta^n \equiv \Delta_0^n$.

Remark 2.1. *For any $f \in s(\mathbb{N}_0;\mathbb{R})$, $l \in \mathbb{N}_0$ we have*

$$\Delta^l f(t) = \sum_{j=0}^{l} \binom{l}{j} (-1)^{l-j} f(t+j), \quad t \in \mathbb{N}_0.$$

We define

$$k^\alpha(n) := \begin{cases} \frac{\Gamma(n+\alpha)}{\Gamma(\alpha)\Gamma(n+1)} & \text{if } \alpha > 0, \quad n \in \mathbb{N}_0, \\ \delta_0(n) & \text{if } \alpha = 0, \end{cases}$$

where $\delta_0(n)$ is the delta function,

$$\delta_0(n) := \begin{cases} 1 & \text{if } n = 0, \\ 0 & \text{if } n \neq 0. \end{cases}$$

The kernel k^α, introduced in [38], appears in many fields of research and has a number of important properties that, in our case, concentrate all the information about the fractional difference operator defined by convolution in (1.2). For a review, see [26, Section 3]. For instance, the semigroup property:

$$(k^\alpha * k^\beta)(n) = k^{\alpha+\beta}(n), \qquad n \in \mathbb{N}_0, \quad \alpha, \beta > 0, \tag{2.1}$$

is frequently used. We recall from [26, Lemma 3.2] the following result.

Lemma 2.2. *For any $\alpha > 0$ and $n \in \mathbb{N}_0$, the following identities hold:*

(i) $\Delta k^\alpha(n) = (\alpha - 1)\dfrac{k^\alpha(n)}{n+1}$.

(ii) $\Delta^2 k^\alpha(n) = (\alpha-2)(\alpha-1)\dfrac{k^\alpha(n)}{(n+1)(n+2)}$.

(iii) $\Delta^3 k^\alpha(n) = (\alpha-3)(\alpha-2)(\alpha-1)\dfrac{k^\alpha(n)}{(n+1)(n+2)(n+3)}$.

The definition of α-th fractional sum on the set $\mathbb{N}_0$ is given by:

Definition 2.3. *For each $\alpha > 0$ and $f \in s(\mathbb{N}_0;\mathbb{R})$, we define the fractional sum of order α as follows:*

$$\Delta^{-\alpha} f(n) := \sum_{j=0}^{n} k^\alpha(n-j) f(j), \quad n \in \mathbb{N}_0.$$

The next concept was proposed in [38], it is analogous to the definition of a fractional derivative in the sense of Riemann-Liouville, see [39].

Definition 2.4. *Let $\alpha > 0$ be given. The $\alpha-$th fractional difference operator is defined by*

$$\Delta^{\alpha} f(n) := \Delta^{m} \Delta^{-(m-\alpha)} f(n), \quad n \in \mathbb{N}_0,$$

where $m-1 < \alpha \leq m,\ m \in \mathbb{N}$.

We recall that the finite convolution $*$ of two sequences u, v where $u \in s(\mathbb{N}_0; \mathbb{C})$ and $v \in s(\mathbb{N}_0; \mathbb{R})$ is defined by

$$(u * v)(n) = \sum_{j=0}^{n} u(n-j)v(j), \quad n \in \mathbb{N}_0.$$

Given $a, b \in \mathbb{R}$, we define the translation (by $a \in \mathbb{R}$) operator $\tau_a : s(\mathbb{N}_a; \mathbb{R}) \to s(\mathbb{N}_0; \mathbb{R})$ by

$$\tau_a f(n) := f(a+n), \quad n \in \mathbb{N}_0.$$

Note that $\tau_a^{-1} = \tau_{-a}$ and $\tau_{a+b} = \tau_a \circ \tau_b = \tau_b \circ \tau_a$.

Lemma 2.5. *[26, Lemma 2.3] Let $f, g \in s(\mathbb{N}_0; \mathbb{R})$ be sequences, then for each $p \in \mathbb{N}$ we have*

$$(f * \tau_p g)(n) = \tau_p (f * g)(n) - \sum_{j=0}^{p-1} \tau_p f(n-j) g(j).$$

We recall that the most commonly used fractional difference operator of order $\nu > 0$ was defined by Atici and Eloe [3, 4, 5]

$$\left(\Delta_a^{\nu} f\right)(t) := \frac{1}{\Gamma(-\nu)} \sum_{s=a}^{t+\nu} (t-s-1)^{\underline{-\nu-1}} f(s),\ t \in \mathbb{N}_{a+N-\nu}, \tag{2.2}$$

where $f \in s(\mathbb{N}_a; \mathbb{R})$, $N \in \mathbb{N}_1$ is the unique integer satisfying $N-1 < \nu < N$, and the map $t \mapsto t^{\underline{\nu}}$ is defined by $t^{\underline{\nu}} := \dfrac{\Gamma(t+1)}{\Gamma(t+1-\nu)}$. In the integer cases of $\nu = N$ we have

$$\Delta_a^N f(t) = \sum_{j=0}^{N} \binom{N}{j} (-1)^{N-j} f(t+j), \quad t \in \mathbb{N}_a. \tag{2.3}$$

In [26, Theorem 4.3] the authors related (2.2) to Definition 2.4 by means of the operator of translation, which allowed to transfer the properties of a sequence u between both definitions and called a *transference principle.* This will be the key tool to prove the theorems stated in the introduction for the fractional difference operator (2.2). In the following, we have extended the formulation of the transference principle in order to include the integer cases $\alpha = N \in \mathbb{N}$, being the proof immediate taking into account (2.3).

Theorem 2.6. *(Transference Principle) Let $N-1 < \alpha \leq N,\ N \in \mathbb{N}$ and $a, \beta \in \mathbb{R}$. For each sequence $f \in s(\mathbb{N}_a; \mathbb{R})$ we have*

$$\tau_{a+N-\alpha} \circ \Delta_a^{\alpha} f = \Delta^{\alpha} \circ \tau_a f,$$

and for each $f \in s(\mathbb{N}_{a+N-\beta}; \mathbb{R})$,

$$\tau_{N-\beta} \circ \Delta_{a+N-\beta}^{\alpha} f = \Delta_a^{\alpha} \circ \tau_{N-\beta} f.$$

In other words, the following diagrams are commutative:

$$\begin{tikzcd} s(\mathbb{N}_a;\mathbb{R}) \arrow[r, "\Delta_a^{\alpha}"] \arrow[d, "\tau_a"] & s(\mathbb{N}_{a+N-\alpha};\mathbb{R}) \arrow[d, "\tau_{a+N-\alpha}"] \\ s(\mathbb{N}_0;\mathbb{R}) \arrow[r, "\Delta^{\alpha}"] & s(\mathbb{N}_0;\mathbb{R}) \end{tikzcd} \qquad \begin{tikzcd} s(\mathbb{N}_{a+N-\beta};\mathbb{R}) \arrow[r, "\Delta_{a+N-\beta}^{\alpha}"] \arrow[d, "\tau_{N-\beta}"] & s(\mathbb{N}_{a+2N-\beta-\alpha};\mathbb{R}) \arrow[d, "\tau_{N-\beta}"] \\ s(\mathbb{N}_a;\mathbb{R}) \arrow[r, "\Delta_a^{\alpha}"] & s(\mathbb{N}_{a+N-\alpha};\mathbb{R}) \end{tikzcd}$$

3. Properties of the operator of fractional difference Δ^α.

In this section we collect some important new properties of the higher order differences Δ^l, for $l \in \mathbb{N}$ and of the $\alpha - th$ fractional difference operator Δ^α, for $\alpha > 0$. These results generalize [26, Proposition 2.9] and [26, Theorems 5.7, 6.11, 6.16, 6.21, 7.8]. These will be essential to prove our main theorems. We begin with the following result.

Proposition 3.1. *For any $a, b \in s(\mathbb{N}_0; \mathbb{R})$ and $l, l_1, l_2 \in \mathbb{N}$ we have*

(*i*)

$$\Delta^l(a * b)(n) = (\Delta^l a * b)(n) + \sum_{j=1}^{l}\sum_{i=0}^{j-1}\binom{l}{j}(-1)^{l-j}a(i)b(n+j-i).$$

(*ii*)

$$\Delta^{l_1+l_2}(a * b)(n) = (\Delta^{l_1} a * \Delta^{l_2} b)(n) + \sum_{j=1}^{l_1+l_2}\sum_{i=0}^{j-1}\binom{l_1+l_2}{j}(-1)^{l_1+l_2-j}a(i)b(n+j-i)$$
$$+ \sum_{j=1}^{l_2}\sum_{i=0}^{j-1}\binom{l_2}{j}(-1)^{l_2-j}[\Delta^{l_1}a(n+j-i)b(i) - \Delta^{l_1}a(i)b(n+j-i)].$$

Proof. (*i*) Note that, by Remark 2.1 and Lemma 2.5 we get for $l \in \mathbb{N}$

$$\begin{aligned}(\Delta^l a * b)(n) &= \sum_{i=0}^{n} b(i)\Delta^l a(n-i) = \sum_{i=0}^{n} b(i)\sum_{j=0}^{l}\binom{l}{j}(-1)^{l-j}a(n+j-i)\\ &= \sum_{i=0}^{n}\sum_{j=0}^{l}\binom{l}{j}(-1)^{l-j}a(n+j-i)b(i) = \sum_{j=0}^{l}\binom{l}{j}(-1)^{l-j}\sum_{i=0}^{n}\tau_j a(n-i)b(i)\\ &= \sum_{j=0}^{l}\binom{l}{j}(-1)^{l-j}(\tau_j a * b)(n), \quad n \in \mathbb{N}_0.\end{aligned}$$

Thus, we have that for $n \in \mathbb{N}_0$,

$$\begin{aligned}\Delta^l(a * b)(n) &= \sum_{j=0}^{l}\binom{l}{j}(-1)^{l-j}(a * b)(n+j) = \sum_{j=1}^{l}\binom{l}{j}(-1)^{l-j}\tau_j(a * b)(n) + \binom{l}{0}(-1)^{l-0}(a * b)(n)\\ &= \sum_{j=1}^{l}\binom{l}{j}(-1)^{l-j}[(\tau_j a * b)(n) + \sum_{i=0}^{j-1}a(i)\tau_j b(n-i)] + (-1)^l(a * b)(n)\\ &= \sum_{j=1}^{l}\binom{l}{j}(-1)^{l-j}(\tau_j a * b)(n) + (-1)^l(a * b)(n) + \sum_{j=1}^{l}\binom{l}{j}(-1)^{l-j}\sum_{i=0}^{j-1}a(i)\tau_j b(n-i)\\ &= \sum_{j=0}^{l}\binom{l}{j}(-1)^{l-j}(\tau_j a * b)(n) + \sum_{j=1}^{l}\sum_{i=0}^{j-1}\binom{l}{j}(-1)^{l-j}a(i)b(n-i+j)\\ &= (\Delta^l a * b)(n) + \sum_{j=1}^{l}\sum_{i=0}^{j-1}\binom{l}{j}(-1)^{l-j}a(i)b(n+j-i).\end{aligned}$$

which proves the result.

(ii) By part (i), we have for $n \in \mathbb{N}_0$,

$$
\begin{aligned}
\Delta^{l_1+l_2}(a*b)(n) &= (\Delta^{l_1+l_2}a*b)(n) + \sum_{j=1}^{l_1+l_2}\sum_{i=0}^{j-1}\binom{l_1+l_2}{j}(-1)^{l_1+l_2-j}a(i)b(n+j-i) \\
&= \Delta^{l_2}(\Delta^{l_1}a*b)(n) - \sum_{j=1}^{l_2}\sum_{i=0}^{j-1}\binom{l_2}{j}(-1)^{l_2-j}\Delta^{l_1}a(i)b(n+j-i) \\
&\quad + \sum_{j=1}^{l_1+l_2}\sum_{i=0}^{j-1}\binom{l_1+l_2}{j}(-1)^{l_1+l_2-j}a(i)b(n+j-i) \\
&= (\Delta^{l_1}a*\Delta^{l_2}b)(n) + \sum_{j=1}^{l_2}\sum_{i=0}^{j-1}\binom{l_2}{j}(-1)^{l_2-j}\Delta^{l_1}a(n+j-i)b(i) \\
&\quad - \sum_{j=1}^{l_2}\sum_{i=0}^{j-1}\binom{l_2}{j}(-1)^{l_2-j}\Delta^{l_1}a(i)b(n+j-i) \\
&\quad + \sum_{j=1}^{l_1+l_2}\sum_{i=0}^{j-1}\binom{l_1+l_2}{j}(-1)^{l_1+l_2-j}a(i)b(n+j-i) \\
&= (\Delta^{l_1}a*\Delta^{l_2}b)(n) + \sum_{j=1}^{l_1+l_2}\sum_{i=0}^{j-1}\binom{l_1+l_2}{j}(-1)^{l_1+l_2-j}a(i)b(n+j-i) \\
&\quad + \sum_{j=1}^{l_2}\sum_{i=0}^{j-1}\binom{l_2}{j}(-1)^{l_2-j}[\Delta^{l_1}a(n+j-i)b(i) - \Delta^{l_1}a(i)b(n+j-i)].
\end{aligned}
$$

$\square$

Remark 3.2. *In particular, from (i), for $l-1<\alpha<l$, $l \in \mathbb{N}$, we have the following useful identities*

$$\Delta^\alpha u(n) = (k^{l-\alpha}*\Delta^l u)(n) + \sum_{j=1}^{l}\sum_{i=0}^{j-1}\binom{l}{j}(-1)^{l-j}u(i)k^{l-\alpha}(n+j-i), \quad n \in \mathbb{N}_0.$$

taking $a(n) := u(n)$ and $b(n) := k^{l-\alpha}(n)$, and

$$\Delta^\alpha u(n) = (\Delta^l k^{l-\alpha}*u)(n) + \sum_{j=1}^{l}\sum_{i=0}^{j-1}\binom{l}{j}(-1)^{l-j}k^{l-\alpha}(i)u(n+j-i), \quad n \in \mathbb{N}_0.$$

taking $b(n) := u(n)$ and $a(n) := k^{l-\alpha}(n)$.

Concerning the composition of two operators, we prove the following result.

Proposition 3.3. *The following properties hold:*

(i) *For any $\alpha > 0$ and $u \in s(\mathbb{N}_0;\mathbb{R})$ we have*

$$\Delta \circ \Delta^\alpha u(n) = \Delta^{\alpha+1}u(n),$$

where $m-1 \le \alpha \le m$, $m \in \mathbb{N}$.

(*ii*) *For any* $\alpha > 0$, $l \in \mathbb{N}_0$ *and* $u \in s(\mathbb{N}_0;\mathbb{R})$ *we have*

$$\Delta^l \circ \Delta^\alpha u(n) = \Delta^{\alpha+l}u(n),$$

where $m-1 \le \alpha \le m$, $m \in \mathbb{N}$.

(*iii*) *For any* $\alpha > 0$ *and* $u \in s(\mathbb{N}_0;\mathbb{R})$ *we have*

$$\Delta^\alpha \circ \Delta u(n) = \Delta^{\alpha+1}u(n) - \Delta^m k^{m-\alpha}(n+1)u(0),$$

where $m-1 \le \alpha \le m$, $m \in \mathbb{N}$.

(*iv*) *For any* $\alpha > 0$, $l \in \mathbb{N}$ *and* $u \in s(\mathbb{N}_0;\mathbb{R})$ *we have*

$$\Delta^\alpha \circ \Delta^l u(n) = \Delta^{\alpha+l}u(n) - \sum_{j=0}^{l-1} \Delta^{m+j}k^{m-\alpha}(n+1)\Delta^{l-1-j}u(0),$$

where $m-1 \le \alpha \le m$, $m \in \mathbb{N}$.

(*v*) *For any* $\alpha, \beta > 0$ *and* $u \in s(\mathbb{N}_0;\mathbb{R})$ *we have*

$$\Delta^\beta \circ \Delta^\alpha u(n) = \Delta^{\alpha+\beta}u(n) - \sum_{j=0}^{l-1} \Delta^{m+j}k^{m-\beta}(n+1)\Delta^{l-1-j}(k^{l-\alpha} * u)(0),$$

where $m-1 \le \beta \le m$, $l-1 < \alpha \le l$, $m, l \in \mathbb{N}$, *and* $m+l-1 < \alpha+\beta \le m+l$.

(*vi*) *For any* $\alpha, \beta > 0$ *and* $u \in s(\mathbb{N}_0;\mathbb{R})$ *we have*

$$\Delta^\beta \circ \Delta^\alpha u(n) = \Delta^{\alpha+\beta}u(n+1) - \sum_{j=0}^{l-1} \Delta^{m+j}k^{m-\beta}(n+1)\Delta^{l-1-j}(k^{l-\alpha} * u)(0),$$

where $m-1 \le \beta \le m$, $l-1 < \alpha \le l$, $m, l \in \mathbb{N}$, *and* $m+l-2 < \alpha+\beta \le m+l-1$.

Proof. (*i*) The cases $\alpha = m$ and $\alpha = m-1$ are trivial. If $m-1 < \alpha < m$, then the proof is immediate from the definition. Indeed,

$$\Delta \circ \Delta^\alpha u(n) = \Delta \circ \Delta^m (k^{m-\alpha} * u)(n) = \Delta^{m+1}(k^{m+1-(\alpha+1)} * u)(n) = \Delta^{\alpha+1}u(n).$$

(*ii*) By proceed by induction on l. For $l = 0$ is trivial and $l = 1$ is the previous case (*i*). For $l \in \mathbb{N}_0$, we have $\Delta^l \circ \Delta^\alpha u(n) = \Delta^{\alpha+l}u(n)$, where $m-1 \le \alpha \le m$. Then, for $l+1$ we obtain

$$\Delta^{l+1} \circ \Delta^\alpha u(n) = \Delta \circ (\Delta^l \circ \Delta^\alpha u)(n) = \Delta \circ (\Delta^{\alpha+l}u)(n) = \Delta^{\alpha+l+1}u(n).$$

(*iii*) If $\alpha = m$, then $k^{m-\alpha}(n+1) = k^0(n+1) = 0$, hence we have the result. Note that, $\Delta^m k(n+1) = \Delta^m \mathbf{1} = 0$, where $\mathbf{1}(n) \equiv 1$, thus we get the case $\alpha = m-1$.

Suppose $m-1<\alpha<m$. By Remark 2.1, Lemma 2.5 and the previous property, we obtain

$$\begin{aligned}
\Delta^{\alpha}\circ\Delta u(n) &= \Delta^m(k^{m-\alpha}*\Delta u)(n)=\Delta^m(k^{m-\alpha}*(\tau_1u-u))(n)\\
&= \Delta^m(k^{m-\alpha}*\tau_1u)(n)-\Delta^m(k^{m-\alpha}*u)(n)\\
&= \sum_{j=0}^{m}\binom{m}{j}(-1)^{m-j}(k^{m-\alpha}*\tau_1u)(n+j)-\Delta^m(k^{m-\alpha}*u)(n)\\
&= \sum_{j=0}^{m}\binom{m}{j}(-1)^{m-j}[(k^{m-\alpha}*u)(n+j+1)-k^{m-\alpha}(n+j+1)u(0)]-\Delta^m(k^{m-\alpha}*u)(n)\\
&= \sum_{j=0}^{m}\binom{m}{j}(-1)^{m-j}(k^{m-\alpha}*u)(n+j+1)-\sum_{j=0}^{m}\binom{m}{j}(-1)^{m-j}k^{m-\alpha}(n+j+1)u(0)\\
&\quad-\Delta^m(k^{m-\alpha}*u)(n)\\
&= \Delta^m(k^{m-\alpha}*u)(n+1)-\Delta^m k^{m-\alpha}(n+1)u(0)-\Delta^m(k^{m-\alpha}*u)(n)\\
&= \Delta\circ\Delta^m(k^{m-\alpha}*u)(n)-\Delta^m k^{m-\alpha}(n+1)u(0)\\
&= \Delta\circ\Delta^{\alpha}u(n)-\Delta^m k^{m-\alpha}(n+1)u(0)\\
&= \Delta^{\alpha+1}u(n)-\Delta^m k^{m-\alpha}(n+1)u(0).
\end{aligned}$$

(iv) By induction on l. For $l=1$ is the previous case (iii). For $l\in\mathbb{N}$, we have

$$\Delta^{\alpha}\circ\Delta^{l}u(n)=\Delta^{\alpha+l}u(n)-\sum_{j=0}^{l-1}\Delta^{m+j}k^{m-\alpha}(n+1)\Delta^{l-1-j}u(0),$$

where $m-1\le\alpha\le m$. Then, for $l+1$ we obtain

$$\begin{aligned}
\Delta^{\alpha}\circ\Delta^{l+1}u(n) &= \Delta^{\alpha}\circ\Delta^{l}(\Delta u)(n)=\Delta^{\alpha+l}(\Delta u)(n)-\sum_{j=0}^{l-1}\Delta^{m+j}k^{m-\alpha}(n+1)\Delta^{l-1-j}(\Delta u)(0)\\
&= \Delta^{l}(\Delta^{\alpha}\circ\Delta u)(n)-\sum_{j=0}^{l-1}\Delta^{m+j}k^{m-\alpha}(n+1)\Delta^{l-j}u(0)\\
&= \Delta^{l}[(\Delta^{\alpha+1}u)(n)-\Delta^m k^{m-\alpha}(n+1)u(0)]-\sum_{j=0}^{l-1}\Delta^{m+j}k^{m-\alpha}(n+1)\Delta^{l-j}u(0)\\
&= \Delta^{\alpha+1+l}u(n)-\Delta^{m+l}k^{m-\alpha}(n+1)u(0)-\sum_{j=0}^{l-1}\Delta^{m+j}k^{m-\alpha}(n+1)\Delta^{l-j}u(0)\\
&= \Delta^{\alpha+1+l}u(n)-\sum_{j=0}^{l}\Delta^{m+j}k^{m-\alpha}(n+1)\Delta^{l-j}u(0).
\end{aligned}$$

(v) First we show the property when $m-1<\beta<m$, $l-1<\alpha<l$ and $m+l-1<\alpha+\beta<m+l$. By definition and using the semigroup property of k^{α}, namely: $k^{\alpha}*k^{\beta}=k^{\alpha+\beta}$ for any $\alpha,\beta>0$, as well as the properties (ii) and (iv), we have for $n \in \mathbb{N}_0$,

$$\begin{aligned}
\Delta^\beta \circ \Delta^\alpha u(n) &= \Delta^\beta \circ \Delta^l (k^{l-\alpha} * u)(n)\\
&= \Delta^{\beta+l}(k^{l-\alpha} * u)(n) - \sum_{j=0}^{l-1} \Delta^{m+j} k^{m-\beta}(n+1)\Delta^{l-1-j}(k^{l-\alpha} * u)(0)\\
&= \Delta^l \circ \Delta^\beta (k^{l-\alpha} * u)(n) - \sum_{j=0}^{l-1} \Delta^{m+j} k^{m-\beta}(n+1)\Delta^{l-1-j}(k^{l-\alpha} * u)(0)\\
&= \Delta^l \circ \Delta^m (k^{m-\beta} * k^{l-\alpha} * u)(n) - \sum_{j=0}^{l-1} \Delta^{m+j} k^{m-\beta}(n+1)\Delta^{l-1-j}(k^{l-\alpha} * u)(0)\\
&= \Delta^{m+l}(k^{m+l-(\alpha+\beta)} * u)(n) - \sum_{j=0}^{l-1} \Delta^{m+j} k^{m-\beta}(n+1)\Delta^{l-1-j}(k^{l-\alpha} * u)(0)\\
&= \Delta^{\alpha+\beta} u(n) - \sum_{j=0}^{l-1} \Delta^{m+j} k^{m-\beta}(n+1)\Delta^{l-1-j}(k^{l-\alpha} * u)(0).
\end{aligned}$$

Thus, we obtain first part of the (v). We observe that the following identity is true in general for $m-1 < \beta < m$ and $l-1 < \alpha < l$:

$$\Delta^\beta \circ \Delta^\alpha u(n) = \Delta^{m+l}(k^{m+l-(\alpha+\beta)} * u)(n) - \sum_{j=0}^{l-1} \Delta^{m+j} k^{m-\beta}(n+1)\Delta^{l-1-j}(k^{l-\alpha} * u)(0). \tag{3.1}$$

Now, we study the bordeline cases. If $\beta = m$, by property (ii), we get $\Delta^m \circ \Delta^\alpha u(n) = \Delta^{\alpha+m} u(n)$ for any $l-1 \le \alpha \le l$. Furthermore, $k^{m-m}(n+1) = k^0(n+1) = 0$, thus

$$\Delta^m \circ \Delta^\alpha u(n) = \Delta^{\alpha+m} u(n) - \sum_{j=0}^{l-1} \Delta^{m+j} k^{m-m}(n+1)\Delta^{l-1-j}(k^{l-\alpha} * u)(0) = \Delta^{\alpha+m} u(n).$$

If $\beta = m-1$, applying again property (ii), we obtain $\Delta^{m-1} \circ \Delta^\alpha u(n) = \Delta^{\alpha+m-1} u(n)$ for any $l-1 \le \alpha \le l$. Also, $\Delta^{m+j} k^{m-(m-1)}(n+1) = \Delta^{m+j} k(n+1) = \Delta^{m+j} 1 = 0$, thus we obtain the result analogously to the proof for $\beta = m$.
If $\alpha = l$, by property (iv), we have $\Delta^\beta \circ \Delta^l u(n) = \Delta^{\beta+l} u(n) - \sum_{j=0}^{l-1} \Delta^{m+j} k^{m-\beta}(n+1)\Delta^{l-1-j} u(0)$, for any $m-1 \le \beta \le m$. Moreover, $\Delta^{l-1-j} u(0) = \Delta^{l-1-j}(k^0 * u)(0) = \Delta^{l-1-j}(k^{l-\alpha} * u)(0)$, that shows the result. Moreover, since it is valid for the points $\alpha = l$ and $\beta = m$, we deduce that in particular it is true for $\alpha + \beta = m + l$.

(vi) We proceed analogously to the proof of property (v). We have, by (3.1), for $m-1 \le \beta \le m$ and $l-1 < \alpha \le l$,

$$\Delta^\beta \circ \Delta^\alpha u(n) = \Delta^{m+l}(k^{m+l-(\alpha+\beta)} * u)(n) - \sum_{j=0}^{l-1} \Delta^{m+j} k^{m-\beta}(n+1)\Delta^{l-1-j}(k^{l-\alpha} * u)(0).$$

Suppose $m+l-2 < \alpha+\beta < m+l-1$. Using the fact that $\Delta(k * u)(n) = u(n+1)$, we have

$$\begin{aligned}
\Delta^{m+l}(k^{m+l-(\alpha+\beta)} * u)(n) &= \Delta^{m+l-1} \circ \Delta(k * k^{m+l-1-(\alpha+\beta)} * u)(n)\\
&= \Delta^{m+l-1}(k^{m+l-1-(\alpha+\beta)} * u)(n+1)\\
&= \Delta^{\alpha+\beta} u(n+1), \quad n \in \mathbb{N}_0.
\end{aligned}$$

Therefore,

$$\Delta^{\beta} \circ \Delta^{\alpha} u(n) = \Delta^{\alpha+\beta} u(n+1) - \sum_{j=0}^{l-1} \Delta^{m+j} k^{m-\beta}(n+1) \Delta^{l-1-j}(k^{l-\alpha} * u)(0).$$

Finally, if $\alpha + \beta = m + l - 1$, then $m - 1 < \beta = m + l - 1 - \alpha < m$ and $l - 1 < \alpha < l$. Hence, we have for all $n \in \mathbb{N}_0$

$$\Delta^{m+l}(k^{m+l-(\alpha+\beta)} * u)(n) = \Delta^{m+l}(k^{m+l-(m+l-1)} * u)(n) = \Delta^{m+l-1} u(n+1) = \Delta^{\alpha+\beta} u(n+1).$$

By (3.1), this finishes the proof.

□

Remark 3.4. *Examining the previous proof, we deduce that for any $\alpha, \beta > 0$ with $m - 1 \leq \beta \leq m$, $l - 1 < \alpha \leq l$, $m, l \in \mathbb{N}$, and $u \in s(\mathbb{N}_0; \mathbb{R})$, if $\alpha + \beta = m + l - 1$, then*

$$\Delta^{\beta} \circ \Delta^{\alpha} u(n) = \Delta^{\alpha+\beta} u(n+1) - \sum_{j=0}^{l-1} \Delta^{m+j} k^{m-\beta}(n+1) \Delta^{l-1-j}(k^{l-\alpha} * u)(0), \ \ n \in \mathbb{N}_0.$$

4. Positivity, α-monotonicity and α-convexity.

In this section we recall the definitions of a α-monotone (increasing or decreasing) and α-convex (resp. concave) sequence, providing a geometrical interpretation. We summarize, and in some cases improve, several theorems from [26, Sections 5 and 6]. We also give new examples showing the necessity of conditions. We begin recalling the following definition.

Definition 4.1. *[7, Definitions 2.3 and 2.4] Let $\alpha \geq 0$ be given. We say that a sequence $u \in s(\mathbb{N}_0, \mathbb{R})$ is $\alpha-$increasing (respectively, decreasing) if*

$$u(n+1) \geq \alpha u(n) \tag{4.1}$$

(respectively, $u(n+1) \leq \alpha u(n)$) for all $\mathbb{N}_0$.

Remark 4.2. *Iterating (4.1), we observe that each α-increasing sequence must satisfy:*

$$u(n) \geq \alpha^n u(0), \quad n \in \mathbb{N}_0.$$

We conclude that if a sequence u is $\alpha-$increasing then their graph lies above the graph of the sequence $M_\alpha(n) := \alpha^n u(0)$. In the Figure 2, assuming $u(0) = 1$, we have drawn the behavior of the sequence $M_\alpha(n)$ for different values of α. In particular, observe that the graph of each increasing sequence lies above the graph of the constant sequence $M_1 \equiv 1$.

Now, let $0 < \alpha < 1$ and note that any increasing sequence that satisfy $u(0) \geq 0$ must be positive and α-increasing. However, the converse is not necessarily true, i.e. an α-increasing sequence could be decreasing for $\alpha < 1$.

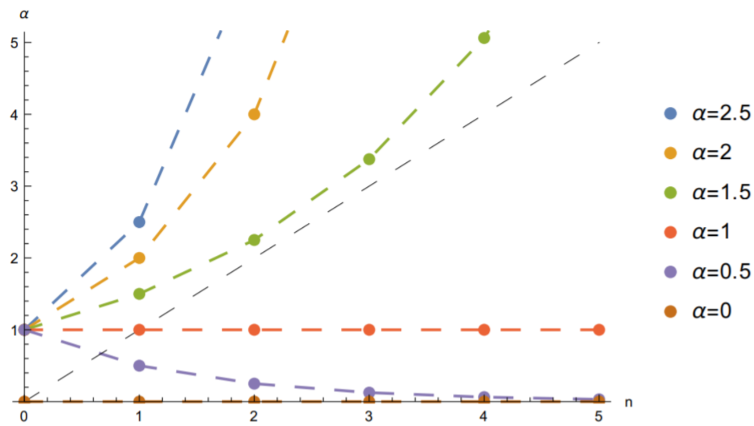

Figure 2: α- monotone increasing sequences

We now recall the following theorem that give us conditions to ensure positivity and α-monotonicity in the closed interval $0 \leq \alpha \leq 1$.

Theorem 4.3. *[26, Theorem 5.4] Let $0 \leq \alpha \leq 1$ and $u \in s(\mathbb{N}_0;\mathbb{R})$ be a given sequence. Suppose that*

(i) $(\Delta^\alpha u)(n) \geq 0$ *for all* $n \in \mathbb{N}_0$,

(ii) $u(0) \geq 0$.

Then u is positive and α-increasing on $\mathbb{N}_0$.

The following example shows that the condition $u(0) \geq 0$ is necessary for positivity.

Example 4.4. *Let $\gamma < \alpha < 1$ and define $u(n) := -\gamma^n,\ 0 < \gamma < 1,\ n \in \mathbb{N}_0$. Then*

- $(\Delta^\alpha u)(n) \geq 0$
- *u is negative.*

Indeed, by part (i) in Proposition 3.1 with $a := k^{1-\alpha}, b := u,\ l = 1$ and part (i) in Lemma 2.2, we obtain the identities

$$(\Delta^\alpha u)(n) = (\Delta k^{1-\alpha} * u)(n) + u(n+1) = -\alpha \sum_{j=0}^{n} \frac{k^{1-\alpha}(n-j)}{n-j+1} u(j) + u(n+1), \quad n \in \mathbb{N}_0.$$

Hence,

$$\begin{aligned}(\Delta^\alpha u)(n) &= -\alpha \sum_{j=0}^{n-1} \frac{k^{1-\alpha}(n-j)}{n-j+1} u(j) + (-\alpha)u(n) + u(n+1)\\ &= \alpha \sum_{j=0}^{n-1} \frac{k^{1-\alpha}(n-j)}{n-j+1} \gamma^j + \gamma^n(\alpha-\gamma) \geq 0,\end{aligned}$$

proving the claim.

Remark 4.5. *Actually, a converse for Theorem 4.3 holds: If $u(0) \geq 0$ and u is increasing (hence positive), then $(\Delta^\alpha u)(n) \geq 0$ for all $n \in \mathbb{N}_0$. This follows immediately from Remark 3.2 with $l = 1$ which asserts*

$$\Delta^\alpha u(n) = (k^{1-\alpha} * \Delta u)(n) + k^{1-\alpha}(n+1)u(0), \quad n \in \mathbb{N}_0, \quad 0 < \alpha < 1.$$

In the following theorem we show sufficient conditions to deduce the positivity and $(\alpha+\beta)-$monotonicity on $\mathbb{N}_0$ of a real sequence u in the region $\mathcal{R} := \{(\alpha,\beta) \in [0,1]\times[0,1] : 0 \le \alpha+\beta \le 1\}$. We remark that it was initially proved in [26, Theorem 5.8] for the open sector $\{(\alpha,\beta) \in (0,1)\times[0,1) : 0 < \alpha+\beta < 1\}$, however we can easily prove that we can extend the same result to the border of the sector.

Theorem 4.6. *Let $(\alpha,\beta) \in \mathcal{R}$ and $u \in s(\mathbb{N}_0;\mathbb{R})$ be a given sequence. Assume that*

(i) $\big(\Delta^\beta \circ \Delta^\alpha u\big)(n) \ge \dfrac{\beta}{2}(1-\beta)u(0)$, *for all $n \in \mathbb{N}_0$;*
(ii) $u(1) \ge (\alpha+\beta)u(0)$;
(iii) $u(0) \ge 0$.

Then u is positive and $(\alpha+\beta)$-increasing on $\mathbb{N}_0$.

Proof. In case $\alpha+\beta = 1$, by Proposition 3.3 part (vi), we have the identity

$$(\Delta^{1-\alpha} \circ \Delta^\alpha u)(n) = \Delta u(n+1) - \Delta k^\alpha(n+1)u(0),$$

and then the proof follows analogously to [26, Theorem 5.8]. We observe that even the case $\alpha = 0$ is true by (i), because $\Delta^\beta \circ \Delta^0 u(n) = \Delta^\beta u(n) \ge 0$, and the proof follows from Theorem 4.3. In conclusion, Theorem 4.6 holds for $(\alpha,\beta) \in \mathcal{R}$. ☐

The following example shows that the hypothesis (ii) in Theorem 4.6 is necessary in order to conclude positivity.

Example 4.7. *Define the sequence $u : \mathbb{N}_0 \to \mathbb{R}$ by $u(0) = 0$ and $u(n) = -2^{-n}$, $n \in \mathbb{N}$. It is clear that $u(0) = 0$ and negative. For $\frac{1}{2} < \alpha+\beta < 1$, note that by Proposition 3.3, part (vi), with $l = m = 1$ we have the identity*

$$(\Delta^\beta \circ \Delta^\alpha u)(n) = (\Delta^{\alpha+\beta}u)(n+1) - \Delta k^{1-\beta}(n+1)u(0) = (\Delta^{\alpha+\beta}u)(n+1),$$

where by Proposition 3.1 part (i) with $a := k^{1-(\alpha+\beta)}, b := u$, $l = 1$ and Lemma 2.2 part (i), we obtain

$$\begin{aligned}(\Delta^{\alpha+\beta}u)(n+1) &= (\Delta k^{1-(\alpha+\beta)} * u)(n+1) + u(n+2)\\ &= -(\alpha+\beta)\sum_{j=0}^{n} \frac{k^{1-(\alpha+\beta)}(n+1-j)}{n+2-j}u(j) - (\alpha+\beta)u(n+1) + u(n+2)\\ &= -(\alpha+\beta)\sum_{j=0}^{n} \frac{k^{1-(\alpha+\beta)}(n+1-j)}{n+2-j}u(j) + 2^{-(n+2)}\left[2(\alpha+\beta)-1\right] \ge 0.\end{aligned}$$

Therefore $(\Delta^\beta \circ \Delta^\alpha u)(n) \ge 0$ and thus (i) in Theorem 4.6 is verified. However, we have $u(1) = -\frac{1}{2} < 0 = (\alpha+\beta)u(0)$.

We next recall the following notion introduced in [26].

Definition 4.8. *Let $\alpha \ge 1$. We say that a sequence $u \in s(\mathbb{N}_0,\mathbb{R})$ is α-convex (respectively, α-concave) if*

$$u(n+2) - \alpha u(n+1) + (\alpha-1)u(n) \ge 0, \quad n \in \mathbb{N}_0, \tag{4.2}$$

(respectively ≤ 0).

Note that, when $\alpha = 2$ we recover the geometrical notion of convexity for a sequence, and when $\alpha = 1$ the concept of monotonicity (increasing) on the set $\mathbb{N}$. It is interesting to observe the following counterpart of Remark 4.2, which is also new.

Remark 4.9. *If $u \in s(\mathbb{N}_0, \mathbb{R})$ is α-convex then for each $\alpha \neq 2$*

$$u(n) \geq \left[\frac{(\alpha-1)^n - 1}{\alpha - 2}\right](u(1) - u(0)) + u(0), \quad n \in \mathbb{N}_0,$$

and

$$u(n) \geq n(u(1) - u(0)) + u(0), \quad n \in \mathbb{N}_0,$$

in case $\alpha = 2$. Indeed, we note that u is α-convex if and only if $\Delta u(n+1) \geq (\alpha-1)\Delta u(n)$, $n \in \mathbb{N}_0$. Iterating, we obtain

$$u(n+1) \geq (\alpha-1)^n(u(1) - u(0)) + u(n), \; n \in \mathbb{N}_0. \tag{4.3}$$

Thus, iterating again we arrive at

$$u(n) \geq \left[\sum_{j=0}^{n-1}(\alpha-1)^j\right](u(1) - u(0)) + u(0),$$

and hence the conclusion follows.

Remark 4.10. *If a sequence u is $\alpha-$convex, then its graph lies above the graph of the sequence $C_\alpha(n) := \left[\frac{1-(\alpha-1)^n}{2-\alpha}\right](u(1)-u(0))+u(0)$ for $\alpha \neq 2$ and above of the graph of the sequence $C_2(n) = n(u(1)-u(0))+u(0)$ in case $\alpha = 2$. Assuming $u(0) = 0, u(1) = 1$, the behavior of the sequence $C_\alpha(n)$ for different values of α is drawn in Figure 3. Since a sequence u is 2-convex iff u is convex, we observe that the graph of each convex sequence lies above the graph of the sequence $C_2(n) = n$. Also, we observe that an α-convex sequence could be geometrically concave for $1 < \alpha < 2$.*

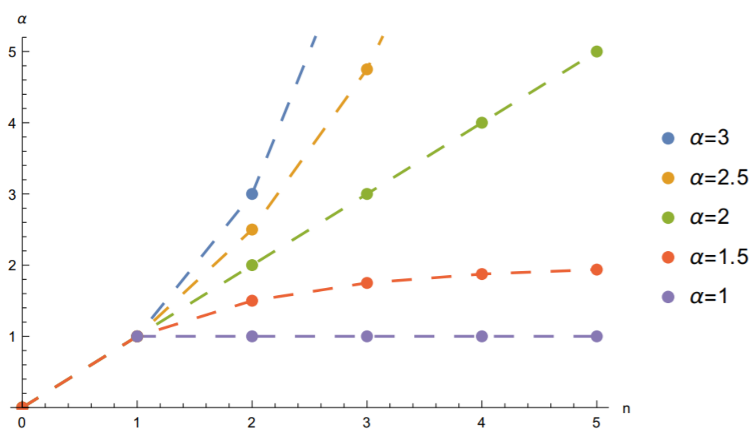

Figure 3: α-convex with $u(0) = 0$ and $u(1) = 1$

Conditions to obtain positivity, monotonicity and $\alpha-$convexity in the closed interval $1 \leq \alpha \leq 2$ are given in the following theorem.

Theorem 4.11. *[26, Theorem 6.3] Let $1 \leq \alpha \leq 2$ and $u \in s(\mathbb{N}_0; \mathbb{R})$ be given and assume that*

(i) $\big(\Delta^\alpha u\big)(n) \geq 0$, *for all* $n \in \mathbb{N}_0$;
(ii) $u(1) \geq \alpha u(0)$;
(iii) $u(0) \geq 0$.

Then u is positive, increasing and α-convex on $\mathbb{N}_0$.

Remark 4.12. *A partial converse of the previous theorem can be established by taking into account the following identity, valid for $1 \le \alpha < 2$, and that follows from Remark 3.2 with $l = 2$:*

$$\Delta^\alpha u(n) = (\Delta^2 k^{2-\alpha} * u)(n) + u(n+2) - \alpha u(n+1), \quad n \in \mathbb{N}_0,$$

which proves that if $u(0) \ge 0$ and u is α-increasing on $\mathbb{N}_0$ (and hence, positive) then $\Delta^\alpha u(n) \ge 0$.

Remark 4.13. *A second partial converse of Theorem 4.11 can be established using now the following identity, that again follows from Remark 3.2 with $l = 2$:*

$$\Delta^\alpha u(n) = (k^{2-\alpha} * \Delta^2 u)(n) + k^{2-\alpha}(n+1)(u(1) - u(0)) + k^{2-\alpha}(n+2)u(0), \quad n \in \mathbb{N}_0.$$

It proves that if $u(0) \ge 0$, $u(1) \ge \alpha u(0)$ (and therefore $u(1) \ge \alpha u(0) \ge u(0)$) and u is convex on $\mathbb{N}_0$ (i.e. 2-convex) then $\Delta^\alpha u(n) \ge 0$.

The following example shows that the condition $u(1) \ge \alpha u(0)$ in Theorem 4.11 is necessary for a sequence to be monotone increasing.

Example 4.14. *Define the sequence $u : \mathbb{N}_0 \to \mathbb{R}$ by $u(n) = \gamma^n$, $0 < \gamma < 1$. Assume that $\frac{1+2\gamma+\sqrt{1+4\gamma-4\gamma^2}}{2} \le \alpha < 2$. The following statements are true.*

- *$(\Delta^\alpha u)(n) \ge 0$, for all $n \in \mathbb{N}_0$;*
- *$u(0) \ge 0$.*
- *u is positive and decreasing.*

Indeed, it is clear that u is such that $u(0) \ge 0$ and is positive and decreasing. Next, observe that by Proposition 3.1 part (i), with $a := k^{2-\alpha}, b := u$, $l = 2$, we obtain for $n \in \mathbb{N}_0$:

$$\begin{aligned}
\Delta^\alpha u(n) &= (\Delta^2 k^{2-\alpha} * u)(n) + \sum_{j=1}^{2} \sum_{i=0}^{j-1} \binom{2}{j} (-1)^{2-j} k^{2-\alpha}(i) u(n+j-i) \\
&= (\Delta^2 k^{2-\alpha} * u)(n) + u(n+2) - \alpha u(n+1) \\
&= \sum_{j=0}^{n-1} \Delta^2 k^{2-\alpha}(n-j) u(j) + \Delta^2 k^{2-\alpha}(0) u(n) + u(n+2) - \alpha u(n+1) \\
&= \sum_{j=0}^{n-1} \Delta^2 k^{2-\alpha}(n-j) \gamma^j + \frac{\alpha(\alpha-1)}{2} \gamma^n + \gamma^{n+2} - \alpha \gamma^{n+1}.
\end{aligned}$$

By Lemma 2.2 part (ii), we have that $\Delta^2 k^{2-\alpha}(n) \ge 0$ for all $n \in \mathbb{N}_0$. Thus,

$$\Delta^\alpha u(n) \ge \frac{\alpha(\alpha-1)}{2} \gamma^n + \gamma^{n+2} - \alpha \gamma^{n+1} = \frac{\gamma^n}{2} [\alpha^2 - \alpha(1+2\gamma) + 2\gamma^2] \ge 0,$$

because $\frac{1+2\gamma+\sqrt{1+4\gamma-4\gamma^2}}{2} \le \alpha < 2$. This proves the claim. On the other hand, we also have $u(1) = \gamma < \alpha = \alpha u(0)$. It shows that the condition $u(1) \ge \alpha u(0)$ in Theorem 4.11 is necessary.

Observe that as γ goes from 0 to 1 the function $\frac{1+2\gamma+\sqrt{1+4\gamma-4\gamma^2}}{2}$ goes from 1 to 2, respectively.

Remark 4.15. *Let $a \in \mathbb{R}$, and $v(n) := \tau_{-a}u(n)$, $n \in \mathbb{N}_0$ where $u(n) = \gamma^n$, $0 < \gamma < 1$. Assume that $\frac{1+2\gamma+\sqrt{1+4\gamma-4\gamma^2}}{2} \leq \alpha < 2$. By Theorem 2.6 and Example 4.14, we have*

$$\Delta_a^\alpha v(t) = (\tau_{a+2-\alpha} \circ \Delta_a^\alpha v)(n) = (\tau_{a+2-\alpha} \circ \Delta_a^\alpha \circ \tau_{-a} u)(n) = \Delta^\alpha u(n) \geq 0, \quad t := n + a + 2 - \alpha.$$

Therefore, we conclude that $\Delta_a^\alpha v(t) \geq 0$ for all $t \in \mathbb{N}_{a+2-\alpha}$, and $v(a) = u(0) \geq 0$. Also, v is decreasing if u is decreasing. Moreover $v(a+1) = u(1) = \gamma < 1 < \alpha = \alpha u(0) = \alpha v(a)$. It follows that the condition $v(a+1) \geq \alpha v(a)$ in Theorem 1.4 is necessary in order to have the conclusion.

Note that this example generalizes [16, Example 2.4] where the authors proved that for $v(t) = 2^{-t}$ and $\frac{2+\sqrt{2}}{2} < \alpha < 2$ they have $\Delta_0^\alpha v(t) \geq 0$ for all $t \in \mathbb{N}_{2-\alpha}$.

In the following theorem we summarize, improve and extend [26, Theorems 6.12, 6.17, 6.22]. Indeed, compared with [26], we have included the borders of the regions and added a new condition, namely $(\Delta^{\alpha+\beta}u)(0) \geq 0$, in order to ensure positivity, monotonicity and $(\alpha+\beta)-$convexity on $\mathbb{N}_0$ of a real valued sequence u in the region $\mathcal{M} := \{(\alpha, \beta) \in [0,2] \times [0,2] : 1 \leq \alpha + \beta \leq 2\}$. We will consider the following subsets of $\mathcal{M}$ (see also Figure 1):

$$\mathcal{M}_1 := \{(\alpha, \beta) \in [0,2] \times [0,2] : 0 \leq \alpha \leq 1, 1 \leq \beta \leq 2, 1 \leq \alpha + \beta \leq 2\},$$

$$\mathcal{M}_2 := \{(\alpha, \beta) \in [0,2] \times [0,2] : 0 \leq \alpha \leq 1, 0 \leq \beta \leq 1, 1 \leq \alpha + \beta \leq 2\},$$

$$\mathcal{M}_3 := \{(\alpha, \beta) \in [0,2] \times [0,2] : 1 \leq \alpha \leq 2, 0 \leq \beta \leq 1, 1 \leq \alpha + \beta \leq 2\}.$$

Theorem 4.16. *Suppose that $u \in s(\mathbb{N}_0; \mathbb{R})$, and*

(*i*)

$$(\Delta^\beta \circ \Delta^\alpha u)(n) \geq \begin{cases} 0 & \text{if } (\alpha, \beta) \in \mathcal{M}_1, \\ \frac{\beta}{2}(1-\beta)u(0) & \text{if } (\alpha, \beta) \in \mathcal{M}_2, \\ \frac{\beta}{2}(1-\beta)[u(1) - (\alpha-1)u(0)] & \text{if } (\alpha, \beta) \in \mathcal{M}_3; \end{cases} \tag{4.4}$$

(*ii*) $u(2) \geq (\alpha+\beta)u(1) - \frac{(\alpha+\beta)(\alpha+\beta-1)}{2}u(0)$;

(*iii*) $u(1) \geq (\alpha+\beta)u(0)$;

(*iv*) $u(0) \geq 0$.

Then u is positive, increasing and $(\alpha+\beta)$-convex on $\mathbb{N}_0$.

Proof. We divide the proof in the following cases:

$\mathcal{M}_1$: $(\alpha, \beta) \in \mathcal{M}_1$. We have that the case $\alpha = 0$ is true by (*i*), because $\Delta^\beta \circ \Delta^0 u(n) = \Delta^\beta u(n) \geq 0$. Hence we can apply Theorem 4.11 for $\beta \in [1, 2]$.

In other case, by Proposition 3.3 part (*vi*), with $l = 1$, $m = 2$, we obtain

$$(\Delta^{\alpha+\beta}u)(n+1) = (\Delta^\beta \circ \Delta^\alpha u)(n) + (\Delta^2 k^{2-\beta})(n+1)u(0). \tag{4.5}$$

Moreover, by Lemma 2.2 part (*ii*), we have $(\Delta^2 k^{2-\beta})(n+1) \geq 0$. Thus, by (*iii*), $(\Delta^2 k^{2-\beta})(n+1)u(0) \geq 0$. Then, by (*i*), we have that

$$(\Delta^{\alpha+\beta}u)(n) \geq 0, \text{ for all } n \in \mathbb{N}.$$

Therefore by hypothesis (*ii*), (*iii*), (*iv*) and Theorem 4.11, the conclusion follows.

$\mathcal{M}_2$: $(\alpha, \beta) \in \mathcal{M}_2$. Note that $k^\gamma(n) \geq 0$ and is decreasing for any $0 < \gamma < 1$ fixed. Then we have

$$\Delta k^\gamma(n+1) = \frac{\gamma - 1}{n+2}k^\gamma(n+1) \geq \frac{\gamma-1}{2}k^\gamma(1) = \frac{\gamma-1}{2}\gamma. \tag{4.6}$$

Since $0 \le 1-\beta \le 1$, by hypothesis (iv) and (4.6), we obtain

$$\Delta k^{1-\beta}(n+1)u(0) \ge -\frac{\beta}{2}(1-\beta)u(0). \tag{4.7}$$

If $\alpha+\beta = 1$ then, by Remark 3.4 with $l = m = 1$, (4.7) and hypothesis (i), we get $\Delta u(n+1) = \Delta^\beta \circ \Delta^\alpha u(n) + \Delta k^{1-\beta}(n+1)u(0) \ge 0$, i.e., u is 1-convex = monotone increasing on $n \in \mathbb{N}$. But, by hypothesis (iii), it is also on $n \in \mathbb{N}_0$.

In other case, by Proposition 3.3 part (v), with $l = m = 1$, we obtain

$$(\Delta^{\alpha+\beta}u)(n) = (\Delta^\beta \circ \Delta^\alpha u)(n) + (\Delta k^{1-\beta})(n+1)u(0), \quad n \in \mathbb{N}_0. \tag{4.8}$$

Therefore, by hyphotesis (i) and (4.6), we have $(\Delta^{\alpha+\beta}u)(n) \ge (\Delta^\beta \circ \Delta^\alpha u)(n) - \frac{\beta}{2}(1-\beta)u(0) \ge 0$. Then, $(\Delta^{\alpha+\beta}u)(n) \ge 0$ on $\mathbb{N}_0$. Using hypothesis (iii), (iv), and Theorem 4.11, the conclusion follows.

$\mathcal{M}_3$: $(\alpha,\beta) \in \mathcal{M}_3$. If $\alpha = 1$, by Proposition 3.3 part (iii), with $l = 2$, $m = 1$, hypothesis (i) and (4.6), we have $\Delta^{\beta+1}u(n) = \Delta^\beta \circ \Delta u(n) + \Delta k^{1-\beta}(n+1)u(0) \ge 0$. Hence we can apply Theorem 4.11 for $\beta+1 \in [1,2]$.

In other case, by Proposition 3.3 part (vi), with $l = 2$, $m = 1$, we have

$$(\Delta^{\alpha+\beta}u)(n+1) = (\Delta^\beta \circ \Delta^\alpha u)(n) + (\Delta^2 k^{1-\beta})(n+1)u(0) + (\Delta k^{1-\beta})(n+1)\Delta^{\alpha-1}u(0). \tag{4.9}$$

Since $\Delta^2 k^{1-\beta}(n+1) \ge 0$ and $\Delta k^{1-\beta}(n+1) \ge -\frac{\beta}{2}(1-\beta)$, by Lemma 2.2 part (ii) and (4.6) respectively. Then by the above identity and the hypothesis (i) and (iv), we obtain

$$\begin{aligned}\big(\Delta^{\alpha+\beta}u\big)(n+1) &\ge \big(\Delta^\beta \circ \Delta^\alpha u\big)(n) + \Delta k^{1-\beta}(n+1)\Delta^{\alpha-1}u(0),\\ &\ge \big(\Delta^\beta \circ \Delta^\alpha u\big)(n) - \frac{\beta}{2}(1-\beta)\Delta^{\alpha-1}u(0).\end{aligned}$$

Hence $(\Delta^{\alpha+\beta}u)(n) \ge 0$ on $\mathbb{N}$. Using the hypothesis and Theorem 4.11 we obtain the conclusion.

□

Remark 4.17. *In Theorem 4.16, for $1 < \alpha < 2$, we have $\Delta^{\alpha-1}u(0) = u(1) - (\alpha-1)u(0)$. Analogously, hypothesis (ii) can be rewritten only in terms of the positivity of $(\Delta^{\alpha+\beta}u)(0)$ because of the following identity*

$$\frac{(\alpha+\beta)(\alpha+\beta-1)}{2}u(0) - (\alpha+\beta)u(1) + u(2) = (\Delta^{\alpha+\beta}u)(0).$$

Remark 4.18. *Note that if $\beta = 1$ the right hand side in (4.4) is exactly the same in the regions $\mathcal{M}_1$ and $\mathcal{M}_2$, and if $\alpha = 1$ then, again, the right hand side in (4.4) of Theorem 4.16 is the same in the regions $\mathcal{M}_2$ and $\mathcal{M}_3$. This means that the given conditions allows a continuous transition between a region and other, without jumps.*

The following example shows that the condition $u(1) \ge (\alpha+\beta)u(0)$ in Theorem 4.16 is necessary for a sequence to be positive and monotone increasing.

Example 4.19. *Define the sequence $u : \mathbb{N}_0 \to \mathbb{R}$ by $u(n) = \gamma^n - 1$, $0 < \gamma < 1$. Assume that $\gamma+1 < \alpha+\beta < 2$. The following statements are true.*

(i) $(\Delta^\beta \circ \Delta^\alpha u)(n) \ge \begin{cases} 0 & \text{if } (\alpha,\beta) \in \mathcal{M}_1. \\ 0 & \text{if } (\alpha,\beta) \in \mathcal{M}_2. \\ \frac{\beta}{2}(1-\beta)u(1) & \text{if } (\alpha,\beta) \in \mathcal{M}_3. \end{cases}$

(ii) $u(2) \geq (\alpha+\beta)u(1)$

(iii) $u(0) = 0$.

(iv) u *is negative and decreasing.*

Indeed, it is clear that u *is such that* $u(0) = 0$ *and is negative and decreasing. Also,* $u(2) = (\gamma+1)(\gamma-1) \geq (\alpha+\beta)(\gamma-1) = (\alpha+\beta)u(1)$. *This proves* $(ii), (iii)$ *and* (iv). *We will prove that* (i) *holds. In fact, observe that by Proposition 3.1 part* (ii) *with* $a := u, b := k^{2-\alpha-\beta}$, $l_1 = l_2 = 1$, *we obtain for* $n \in \mathbb{N}_0$

$$
\begin{aligned}
\Delta^{\alpha+\beta}u(n) &= (\Delta u * \Delta k^{2-\alpha-\beta})(n) - 2u(0)k^{2-\alpha-\beta}(n+1) + u(0)k^{2-\alpha-\beta}(n+2) \\
&\quad + u(1)k^{2-\alpha-\beta}(n+1) + \Delta u(n+1)k^{2-\alpha-\beta}(0) - \Delta u(0)k^{2-\alpha-\beta}(n+1) \\
&= (\Delta u * \Delta k^{2-\alpha-\beta})(n) - 2u(0)k^{2-\alpha-\beta}(n+1) + u(0)k^{2-\alpha-\beta}(n+2) \\
&\quad + u(1)k^{2-\alpha-\beta}(n+1) + \Delta u(n+1) - [u(1)-u(0)]k^{2-\alpha-\beta}(n+1) \\
&= \sum_{j=0}^{n} \Delta u(n-j)\Delta k^{2-\alpha-\beta}(j) + \Delta u(n+1) + \Delta k^{2-\alpha-\beta}(n+1)u(0) \\
&= \sum_{j=1}^{n} \Delta u(n-j)\Delta k^{2-\alpha-\beta}(j) + (1-\alpha-\beta)\Delta u(n) \\
&\quad + \Delta u(n+1) + \Delta k^{2-\alpha-\beta}(n+1)u(0).
\end{aligned}
$$

By Lemma 2.2 part (i), *we have that* $\Delta k^{2-\alpha-\beta}(n) \leq 0$ *for all* $n \in \mathbb{N}_0$. *Thus,*

$$
\begin{aligned}
\Delta^{\alpha+\beta}u(n) &\geq (1-\alpha-\beta)\Delta u(n) + \Delta u(n+1) = (1-\alpha-\beta)\gamma^n(\gamma-1) + \gamma^{n+1}(\gamma-1) \\
&= \gamma^n(\gamma-1)[(1-\alpha-\beta)+\gamma] \geq 0,
\end{aligned} \tag{4.10}
$$

because $\gamma + 1 < \alpha + \beta < 2$. *On the other hand, by the identities* (4.5), (4.8), (4.9) *and* $u(0) = 0$, *we have*

$$
(\Delta^\beta \circ \Delta^\alpha u)(n) = \begin{cases} \Delta^{\alpha+\beta}u(n+1) & \text{if } (\alpha,\beta) \in \mathcal{M}_1. \\ \Delta^{\alpha+\beta}u(n) & \text{if } (\alpha,\beta) \in \mathcal{M}_2. \\ \Delta^{\alpha+\beta}u(n+1) - \Delta k^{1-\beta}(n+1)u(1) & \text{if } (\alpha,\beta) \in \mathcal{M}_3. \end{cases}
$$

Moreover, by (4.6), *we have the inequality* $-\Delta k^{1-\beta}(n+1)u(1) \geq \frac{\beta}{2}(1-\beta)u(1)$. *This, together with* (4.10), *proves* (i). *On the other hand, we also have* $u(1) = \gamma - 1 < 0 = (\alpha+\beta)u(0)$. *It shows that the condition* $u(1) \geq (\alpha+\beta)u(0)$ *in Theorem 4.16 is necessary.*

5. Monotonicity and convexity.

In this section we improve several results from [26, Section 7] and show, in some cases, new conditions to ensure positivity, monotonicity and convexity.

First studies on convexity of difference operators were performed by Atici and Yaldiz [8] and Baoguo et.al. [9]. The next theorem is a substantial improvement of [26, Theorem 7.1] that now ensure the properties of positivity and monotonicity in the semi-closed interval $2 \leq \alpha < 3$ which were not considered in [26]. It is important to observe that this new result, together with those proved in the previous section, allows us to conclude that the properties of positivity, monotonicity and convexity for a given sequence u have a continuous transition as α increase from 0 to 3. Our proof for convexity uses the new properties of the operator Δ^α established in Section 4.

The connection between the fractional difference operator (2.2) and convexity in the context of sequential fractional differences was first considered by Goodrich [22]. He consider, in relation to Figure 1, the cases of the open sectors $\mathcal{C}_3$, $\mathcal{C}_2$ and $\mathcal{C}_4$, named in the reference [22]: Cases I, II and III, respectively. Sharp convexity results for the sector $\mathcal{C}_2$ can be found in the reference [23].

Theorem 5.1. *Let $2 \le \alpha < 3$ and $u \in s(\mathbb{N}_0;\mathbb{R})$ be given and assume that*

(i) $\Delta^\alpha u(n) \ge 0$*, for all* $n \in \mathbb{N}_0$*;*
(ii) $u(2) \ge \alpha u(1) - \frac{\alpha(\alpha-1)}{2}u(0)$*;*
(iii) $u(1) \ge \alpha u(0)$*;*
(iv) $u(0) \ge 0$*.*

Then u is positive, increasing and convex on $\mathbb{N}_0$.

Proof. If $\alpha = 2$ then by hypothesis *(i)*, $\Delta^2 u(n) \ge 0$, for all $n \in \mathbb{N}_0$, i.e. u is convex on $\mathbb{N}_0$. Now, using the fact that u is convex on $\mathbb{N}_0$, we get $\Delta u(n+1) \ge \Delta u(n)$. By hypothesis *(iii)* and *(iv)* we also have $u(1) \ge u(0)$, then $\Delta u(0) \ge 0$ and

$$\Delta u(n+1) \ge \Delta u(n) \ge ... \ge \Delta u(0) \ge 0.$$

Hence, u is monotone increasing and positive on $\mathbb{N}_0$. Now, we assume $2 < \alpha < 3$. By Remark 3.2 with $l = 3$, we have

$$\begin{aligned}(k^{3-\alpha} * \Delta^3 u)(n) &= \Delta^\alpha u(n) - \sum_{j=1}^{3}\sum_{i=0}^{j-1}\binom{3}{j}(-1)^{3-j}u(i)k^{3-\alpha}(n+j-i)\\ &= \Delta^\alpha u(n) - \sum_{j=1}^{3}\sum_{i=0}^{j-1}\binom{3}{j}(-1)^{3-j}u(i)\tau_{j-i}k^{3-\alpha}(n).\end{aligned}$$

Convolving with $k^{\alpha-2}$ we obtain

$$(k^{\alpha-2} * k^{3-\alpha} * \Delta^3 u)(n) = (k^{\alpha-2} * \Delta^\alpha u)(n) - \sum_{j=1}^{3}\sum_{i=0}^{j-1}\binom{3}{j}(-1)^{3-j}u(i)(k^{\alpha-2} * \tau_{j-i}k^{3-\alpha})(n).$$

Observe that, by Lemma 2.5 and the semigroup property of the kernel k^γ, we get

$$\begin{aligned}(k^{\alpha-2} * \tau_{j-i}k^{3-\alpha})(n) &= (k^{\alpha-2} * k^{3-\alpha})(n+j-i) - \sum_{l=0}^{j-i-1} k^{\alpha-2}(n-l+j-i)k^{3-\alpha}(l)\\ &= 1 - \sum_{l=0}^{j-i-1} k^{\alpha-2}(n-l+j-i)k^{3-\alpha}(l).\end{aligned}$$

Therefore,

$$\begin{aligned}\Delta^2 u(n+1) - \Delta^2 u(0) = (k^{\alpha-2} * \Delta^\alpha u)(n) &- \sum_{j=1}^{3}\sum_{i=0}^{j-1}\binom{3}{j}(-1)^{3-j}u(i)\\ &+ \sum_{j=1}^{3}\sum_{i=0}^{j-1}\binom{3}{j}(-1)^{3-j}u(i)\sum_{l=0}^{j-i-1} k^{\alpha-2}(n-l+j-i)k^{3-\alpha}(l).\end{aligned} \tag{5.1}$$

Note that,

$$\sum_{j=1}^{3}\sum_{i=0}^{j-1}\binom{3}{j}(-1)^{3-j}u(i) = 3u(0) - 3u(0) - 3u(1) + u(0) + u(1) + u(2) = \Delta^2 u(0). \tag{5.2}$$

Also, since for any $\gamma > 0$, $k^\gamma(0) = 1$, $k^\gamma(1) = \gamma$ and $k^\gamma(2) = \frac{\gamma(\gamma+1)}{2}$, we have

$$\begin{aligned}
&\sum_{j=1}^{3}\sum_{i=0}^{j-1}\binom{3}{j}(-1)^{3-j}u(i)\sum_{l=0}^{j-i-1}k^{\alpha-2}(n-l+j-i)k^{3-\alpha}(l)\\
&= 3u(0)k^{\alpha-2}(n+1) - 3[u(0)(k^{\alpha-2}(n+2) + k^{\alpha-2}(n+1)(3-\alpha)) + u(1)k^{\alpha-2}(n+1)]\\
&+ [u(0)(k^{\alpha-2}(n+3) + k^{\alpha-2}(n+2)(3-\alpha) + k^{\alpha-2}(n+1)\frac{(3-\alpha)(4-\alpha)}{2})\\
&+ u(1)(k^{\alpha-2}(n+2) + k^{\alpha-2}(n+1)(3-\alpha)) + u(2)k^{\alpha-2}(n+1)].
\end{aligned} \tag{5.3}$$

Replacing (5.2) and (5.3) in (5.1) we obtain that for $n \in \mathbb{N}_0$,

$$\begin{aligned}
\Delta^2 u(n+1) &= (k^{\alpha-2} * \Delta^\alpha u)(n) + k^{\alpha-2}(n+3)u(0) + k^{\alpha-2}(n+2)[u(1) - \alpha u(0)]\\
&\quad + k^{\alpha-2}(n+1)\left[u(2) - \alpha u(1) + \frac{\alpha(\alpha-1)}{2}u(0)\right].
\end{aligned} \tag{5.4}$$

Using the hypothesis (ii), (iii) and (iv) we conclude from (5.4) that $\Delta^2 u(n) \geq 0$, for all $n \in \mathbb{N}$. On the other hand, using (ii), we have

$$u(2) - \alpha u(1) + \frac{\alpha(\alpha-1)}{2}u(0) = \Delta^2 u(0) - (\alpha-2)u(1) + \frac{(\alpha-2)(\alpha-1)}{2}u(0) \geq 0.$$

Hence, hypothesis (iii) and (iv) show that

$$\begin{aligned}
\Delta^2 u(0) &\geq (\alpha-2)u(1) - \frac{(\alpha-2)(\alpha-1)}{2}u(0)\\
&\geq \left[(\alpha-2)\alpha - \frac{(\alpha-2)(\alpha-1)}{2}\right]u(0) = \frac{(\alpha-2)(\alpha+1)}{2}u(0) \geq 0.
\end{aligned}$$

This proves that $\Delta^2 u(n) \geq 0$ for all $n \in \mathbb{N}_0$ – i.e., u is convex. $\square$

Remark 5.2. *Note that if a sequence u is convex on $\mathbb{N}_0$, and $u(1) > u(0) \geq 0$ then it is positive and increasing.*

Remark 5.3. *We can state the following converse for Theorem 5.1: Suppose that $u(0) \geq 0$, $u(1) \geq \alpha u(0)$ and u is α-convex. Then $\Delta^\alpha u(n) \geq 0$. In fact, we first observe that α-convexity implies $u(n+1) - u(n) \geq (\alpha-1)^n(u(1)-u(0))$ (see the proof of Remark 4.9). Since $(\alpha-1)^n(u(1)-u(0)) = (\alpha-1)^n(u(1)-\alpha u(0)) + (\alpha-1)^{n+1}u(0)$, we obtain in view of the given hypothesis that $\Delta u(n) \geq 0$. On the other hand, from Proposition 3.1 with $a := u, b := k^{3-\alpha}, l_1 = 1, l_2 = 2$ we have that the following identity holds:*

$$\Delta^\alpha u(n) = (\Delta u * \Delta^2 k^{3-\alpha})(n) + \Delta^2 k^{3-\alpha}(n+1)u(0) + (\alpha-1)u(n+1) - \alpha u(n+2) + u(n+3).$$

Since $\Delta^2 k^{3-\alpha}(n) \geq 0$, it then follows from the given hypothesis and the previous identity that $\Delta^\alpha u(n) \geq 0$, as claimed.

The previous Remark, together with Remark 4.13, allows one to conclude that the corresponding analogue for the fractional difference operator Δ^α of the well-known property

$$u \text{ convex } \implies \Delta^2 u(n) \geq 0,$$

for $\alpha \in (1,3)$, could be read as follows:

$$u(0) \geq 0,\ \ u(1) \geq \alpha u(0),\ \ u \text{ convex } \implies \Delta^\alpha u(n) \geq 0 \quad (1 < \alpha \leq 2)$$

and

$$u(0) \geq 0, \;\; u(1) \geq \alpha u(0), \;\; u \;\alpha\text{-convex} \implies \Delta^\alpha u(n) \geq 0 \quad (2 \leq \alpha < 3).$$

Indeed, taking into account that convex = 2-convex, we may conclude that as α increases from 1 to 3 then the geometrical property of convexity change continuously from convexity to α-convexity, which gives a reasonable converse for Theorems 4.11 and 5.1.

The following example shows that the condition $u(2) \geq \alpha u(1) - \frac{\alpha(\alpha-1)}{2}u(0)$ in Theorem 5.1 is necessary for convexity.

Example 5.4. *Define $u : \mathbb{N}_0 \to \mathbb{R}$ by $u(n) = \gamma - \frac{1}{\gamma^{n-1}}$ where $\gamma > 1$ is fixed. Observe that $\frac{3\gamma+2+\sqrt{\gamma^2+4\gamma-4}}{2\gamma} < 3$ and let $\frac{3\gamma+2+\sqrt{\gamma^2+4\gamma-4}}{2\gamma} \leq \alpha < 3$. The following statements are true:*

- *$\Delta^\alpha u(n) \geq 0$, for all $n \in \mathbb{N}_0$;*
- *$u(1) \geq \alpha u(0)$;*
- *$u(0) \geq 0$;*
- *u positive, monotone increasing and concave.*

Indeed, first observe that $u(0) = 0$, and $u(1) = \gamma - 1 > 0$. Also, we have that u is positive and $\Delta u(n) = u(n+1) - u(n) = \frac{\gamma-1}{\gamma^n} \geq 0$, i.e., u is monotone increasing on $\mathbb{N}_0$.

Now, by Proposition 3.1 part (ii) with $a := k^{3-\alpha}$, $b := u$ and $l_1 = 2$, $l_2 = 1$, we obtain for each $n \in \mathbb{N}_0$

$$\begin{aligned}
\Delta^\alpha u(n) &= (\Delta^2 k^{3-\alpha} * \Delta u)(n) + 3k^{3-\alpha}(0)u(n+1) - 3[k^{3-\alpha}(0)u(n+2) + k^{3-\alpha}(1)u(n+1)] \\
&\quad + [k^{3-\alpha}(0)u(n+3) + k^{3-\alpha}(1)u(n+2) + k^{3-\alpha}(2)u(n+1)] + \Delta^2 k^{3-\alpha}(n+1)u(0) \\
&\quad - \Delta^2 k^{3-\alpha}(0)u(n+1) \\
&= (\Delta^2 k^{3-\alpha} * \Delta u)(n) + u(n+3) + u(n+2)[-3 + k^{3-\alpha}(1)] \\
&\quad + u(n+1)[3 - 3k^{3-\alpha}(1) + k^{3-\alpha}(2) - \Delta^2 k^{3-\alpha}(0)] + \Delta^2 k^{3-\alpha}(n+1)u(0) \\
&= \sum_{j=0}^{n} \Delta^2 k^{3-\alpha}(j)\Delta u(n-j) + u(n+3) - \alpha u(n+2) + (\alpha-1)u(n+1) + \Delta^2 k^{3-\alpha}(n+1)u(0) \\
&= \sum_{j=1}^{n} \Delta^2 k^{3-\alpha}(j)\Delta u(n-j) + \Delta^2 k^{3-\alpha}(0)\Delta u(n) + u(n+3) - \alpha u(n+2) + (\alpha-1)u(n+1) \\
&\quad + \Delta^2 k^{3-\alpha}(n+1)u(0) \\
&= \sum_{j=1}^{n} \Delta^2 k^{3-\alpha}(j)\Delta u(n-j) + \frac{(\alpha-1)(\alpha-2)}{2}[u(n+1) - u(n)] + u(n+3) - \alpha u(n+2) \\
&\quad + (\alpha-1)u(n+1) + \Delta^2 k^{3-\alpha}(n+1)u(0) \\
&= \sum_{j=1}^{n} \Delta^2 k^{3-\alpha}(j)\Delta u(n-j) + u(n+3) - \alpha u(n+2) + \frac{\alpha(\alpha-1)}{2}u(n+1) - \frac{(\alpha-1)(\alpha-2)}{2}u(n) \\
&\quad + \Delta^2 k^{3-\alpha}(n+1)u(0) \\
&= \sum_{j=1}^{n} \Delta^2 k^{3-\alpha}(j)\Delta u(n-j) + \frac{\gamma^{n+3}-1}{\gamma^{n+2}} - \alpha\frac{\gamma^{n+2}-1}{\gamma^{n+1}} + \frac{\alpha(\alpha-1)}{2}\frac{\gamma^{n+1}-1}{\gamma^n} - \frac{(\alpha-1)(\alpha-2)}{2}\frac{\gamma^n-1}{\gamma^{n-1}}.
\end{aligned}$$

By Lemma 2.2, part (ii), and $\Delta u(n) \geq 0$, *we have* $\sum_{j=1}^{n} \Delta^2 k^{3-\alpha}(j)\Delta u(n-j) \geq 0$. *Thus, since* $\alpha \in [\frac{3\gamma+2+\sqrt{\gamma^2+4\gamma-4}}{2\gamma}, 3)$, *and from the above, we obtain*

$$\begin{aligned}\Delta^\alpha u(n) &\geq \frac{\gamma^{n+3}-1}{\gamma^{n+2}} - \alpha\frac{\gamma^{n+2}-1}{\gamma^{n+1}} + \frac{\alpha(\alpha-1)}{2}\frac{\gamma^{n+1}-1}{\gamma^n} - \frac{(\alpha-1)(\alpha-2)}{2}\frac{\gamma^n-1}{\gamma^{n-1}} \\ &= \frac{(\gamma^3-\gamma^2)\alpha^2 + (2\gamma+\gamma^2-3\gamma^3)\alpha - 2 + 2\gamma^3}{2\gamma^{n+2}} \geq 0.\end{aligned}$$

However u *is concave, indeed,*

$$\Delta^2 u(n) = u(n+2) - 2u(n+1) + u(n) = \frac{\gamma^{n+2}-1}{\gamma^{n+1}} - 2\frac{\gamma^{n+1}-1}{\gamma^n} + \frac{\gamma^n-1}{\gamma^{n-1}} = -\frac{(\gamma-1)^2}{\gamma^{n+1}} \leq 0.$$

Moreover, $u(2) = \frac{(\gamma-1)(\gamma+1)}{\gamma} < 2(\gamma-1) < \alpha(\gamma-1) = \alpha u(1) - \frac{\alpha(\alpha-1)}{2}u(0)$. *It follows that the condition* $u(2) \geq \alpha u(1) - \frac{\alpha(\alpha-1)}{2}u(0)$ *in Theorem 5.1 is necessary in order to ensure the convexity of the sequence* u.

Finally, observe that $\lim_{\gamma\to 1} \frac{3\gamma+2+\sqrt{\gamma^2+4\gamma-4}}{2\gamma} = 3$, and $\lim_{\gamma\to\infty} \frac{3\gamma+2+\sqrt{\gamma^2+4\gamma-4}}{2\gamma} = 2$. Consequently, the interval where α runs in the above example is better as γ increases.

Remark 5.5. *Let* $a \in \mathbb{R}$, *and* $v(n) := \tau_{-a}u(n)$, $n \in \mathbb{N}_0$ *where* $u(n) = \frac{\gamma^n-1}{\gamma^{n-1}}$, $\gamma > 1$. *Asumme that* $\frac{3\gamma+2+\sqrt{\gamma^2+4\gamma-4}}{2\gamma} \leq \alpha < 3$, *by Theorem 2.6, and Example 5.4, we have for* $t := n + a + 3 - \alpha \in \mathbb{N}_{a+3-\alpha}$

$$\Delta_a^\alpha v(t) = (\tau_{a+3-\alpha} \circ \Delta_a^\alpha \circ \tau_{-a}u)(n) = (\tau_{a+3-\alpha} \circ \Delta_a^\alpha v)(n) = \Delta^\alpha u(n) \geq 0.$$

Therefore, we conclude that $\Delta_a^\alpha v(t) \geq 0$ *for all* $t \in \mathbb{N}_{a+3-\alpha}$, $v(a) = u(0) \geq 0$, *and* $v(a+1) = u(1) \geq \alpha u(0) = \alpha v(a)$. *But* $v(a+2) = u(2) = \frac{(\gamma-1)(\gamma+1)}{\gamma} < \alpha(\gamma-1) = \alpha u(1) - \frac{\alpha(\alpha-1)}{2}u(0) = \alpha v(a+1) - \frac{\alpha(\alpha-1)}{2}v(a)$. *It follows that the condition* $v(a+2) \geq \alpha v(a+1) - \frac{\alpha(\alpha-1)}{2}v(a)$ *in Theorem 1.5 is necessary in order to ensure the convexity of the sequence* v.

The following Theorem widely improves [26, Theorems 7.9, 7.11, 7.13, 7.15, 7.17]. We have included the borders of each region given and we have added a new hypothesis, namely: $(\Delta^{\alpha+\beta}u)(0) \geq 0$, in order to ensure positivity, monotonicity and convexity on $\mathbb{N}_0$ of a real sequence u in the set $\mathcal{C} := \{(\alpha,\beta) \in [0,3]\times[0,3] : 1 \leq \alpha+\beta \leq 2\}$. This allow us to see that all the conditions indicated in the theorem, below, overlap with all the conditions in Theorem 4.16. It implies that all the properties of a sequence u remain valid as the parameters (α,β) move and cross from one band $\mathcal{R}, \mathcal{M}$ or $\mathcal{C}$ to another. See Figure 1.

We will consider the following subregions of $\mathcal{C}$,

$$\begin{aligned}\mathcal{C}_1 &:= \{(\alpha,\beta) \in [0,3]\times[0,3] : 0 \leq \alpha \leq 1, 2 \leq \beta < 3, 2 \leq \alpha+\beta < 3\},\\ \mathcal{C}_2 &:= \{(\alpha,\beta) \in [0,3]\times[0,3] : 0 \leq \alpha \leq 1, 1 \leq \beta \leq 2, 2 \leq \alpha+\beta < 3\},\\ \mathcal{C}_3 &:= \{(\alpha,\beta) \in [0,3]\times[0,3] : 1 \leq \alpha \leq 2, 1 \leq \beta \leq 2, 2 \leq \alpha+\beta < 3\},\\ \mathcal{C}_4 &:= \{(\alpha,\beta) \in [0,3]\times[0,3] : 1 \leq \alpha \leq 2, 0 \leq \beta \leq 1, 2 \leq \alpha+\beta < 3\},\\ \mathcal{C}_5 &:= \{(\alpha,\beta) \in [0,3]\times[0,3] : 2 \leq \alpha < 3, 0 \leq \beta \leq 1, 2 \leq \alpha+\beta < 3\}.\end{aligned}$$

Theorem 5.6. *Suppose that*

(*i*)

$$(\Delta^\beta \circ \Delta^\alpha u)(n) \geq \begin{cases} \beta(\beta-1)(\beta-2)\frac{(3-\beta)}{24}u(0) & \text{if } (\alpha,\beta) \in \mathcal{C}_1.\\ 0 & \text{if } (\alpha,\beta) \in \mathcal{C}_2.\\ 0 & \text{if } (\alpha,\beta) \in \mathcal{C}_3.\\ \frac{\beta}{2}(1-\beta)[u(1)-(\alpha-1)u(0)] & \text{if } (\alpha,\beta) \in \mathcal{C}_4.\\ \frac{\beta}{2}(1-\beta)[u(2)-(\alpha-1)u(1)+\frac{(\alpha-1)(\alpha-2)}{2}u(0)] & \text{if } (\alpha,\beta) \in \mathcal{C}_5.\end{cases} \tag{5.5}$$

(*ii*) $u(3) \geq (\alpha+\beta)u(2) - \frac{1}{2}(\alpha+\beta)(\alpha+\beta-1)u(1) + \frac{1}{6}(\alpha+\beta)(\alpha+\beta-1)(\alpha+\beta-2)u(0)$;

(*iii*) $u(2) \geq (\alpha+\beta)u(1) - \frac{1}{2}(\alpha+\beta)(\alpha+\beta-1)u(0)$;

(*iv*) $u(1) \geq (\alpha+\beta)u(0)$;

(*v*) $u(0) \geq 0$.

Then u is positive, monotone increasing and convex on $\mathbb{N}_0$.

Proof. We divide the proof in the following cases:

$\mathcal{C}_1$: Consider $(\alpha,\beta) \in \mathcal{C}_1$. The case $\alpha = 0$ is true by (i), because $\Delta^\beta \circ \Delta^0 u(n) = \Delta^\beta u(n) \geq 0$. Hence we can apply Theorem 5.1 for $\beta \in [2,3)$. In other cases, by Proposition 3.3 part (vi), with $l=1$, $m=3$, we have

$$\Delta^{\alpha+\beta}u(n+1) = \Delta^\beta \circ \Delta^\alpha u(n) + \Delta^3 k^{3-\beta}(n+1)u(0). \tag{5.6}$$

By part (iii) of Lemma 2.2, we obtain

$$\Delta^3 k^{3-\beta}(n+1) = -\beta(\beta-1)(\beta-2)\frac{k^{3-\beta}(n+1)}{(n+2)(n+3)(n+4)}.$$

Since that $0 < 3-\beta < 1$, we deduce that

$$\Delta^3 k^{3-\beta}(n+1) \geq -\beta(\beta-1)(\beta-2)\frac{k^{3-\beta}(1)}{24}.$$

Thus,

$$\Delta^{\alpha+\beta}u(n+1) \geq \Delta^\beta \circ \Delta^\alpha u(n) - \beta(\beta-1)(\beta-2)\frac{k^{3-\beta}(1)}{24}.$$

Then, by hypothesis (i), we conclude that $\Delta^{\alpha+\beta}u(n+1) \geq 0$. Therefore the claim follows from hypothesis and Theorem 5.1.

$\mathcal{C}_2$: Suppose $(\alpha,\beta) \in \mathcal{C}_2$. If $\alpha+\beta = 2$, then by Remark 3.4, with $l=1$, $m=2$ we have

$$\Delta^2 u(n+1) = (\Delta^{2-\alpha} \circ \Delta^\alpha)u(n) + \Delta^2 k^{1-\alpha}(n+1)u(0).$$

By Lemma 2.2 part (ii), we have $\Delta^2 k^{1-\alpha}(n+1)u(0) \geq 0$. Moreover, by hypothesis (i) we have $\Delta^2 u(n+1) \geq 0$. Thus, by hypothesis (iv) and Remark 5.2, we obtain the claimed conclusions.

In other cases, by Proposition 3.3 part (v), with $l=1$, $m=2$ we have

$$\Delta^\beta \circ \Delta^\alpha u(n) = \Delta^{\alpha+\beta}u(n) - \Delta^2 k^{2-\beta}(n+1)u(0). \tag{5.7}$$

But by Lemma 2.2 part (ii), $\Delta^2 k^{2-\beta}(n+1) \geq 0$. Then, by hypothesis (i)

$$\Delta^{\alpha+\beta}u(n) = \Delta^\beta \circ \Delta^\alpha u(n) + \Delta^2 k^{2-\beta}(n+1)u(0) \geq \Delta^\beta \circ \Delta^\alpha u(n) \geq 0.$$

Thus, $\Delta^{\alpha+\beta}u(n) \geq 0$ and the conclusions follows from hypothesis and Theorem 5.1.

$\mathcal{C}_3$: Assume $(\alpha,\beta) \in \mathcal{C}_3$. If $\alpha = 1$ and $1 \leq \beta \leq 2$ then, by Proposition 3.3 part (iii), with $m=2$, we have

$$\Delta^{\beta+1}u(n) = \Delta^\beta \circ \Delta u(n) + \Delta^2 k^{2-\beta}(n+1)u(0).$$

Thus by hypothesis (i), and since that $\Delta^2 k^{2-\beta}(n+1) \geq 0$, we obtain $\Delta^{\beta+1}u(n) \geq 0$. Hence, we can apply Theorem 5.1 for $\beta \in [1,2]$ and the conclusion follows. In other cases, by Proposition 3.3 part (vi), with $l=m=2$, we have the identity

$$\Delta^{\alpha+\beta}u(n+1) = \Delta^\beta \circ \Delta^\alpha u(n) + \Delta^2 k^{2-\beta}(n+1)\Delta^{\alpha-1}u(0) + \Delta^3 k^{2-\beta}(n+1)u(0). \tag{5.8}$$

Note that, $1+\beta < n+4$ for all $n \in \mathbb{N}_0$, thus $\frac{1+\beta}{n+4} - 1 < 0$, for all $n \in \mathbb{N}_0$. Then, $\beta + \alpha > \alpha + \frac{1+\beta}{n+4} - 1$ and we obtain

$$(\alpha+\beta)u(0) \geq [\alpha - 1 + \frac{1+\beta}{n+4}]u(0).$$

Since $u(1) \geq (\alpha+\beta)u(0)$, then

$$u(1) \geq [\alpha - 1 + \frac{1+\beta}{n+4}]u(0). \tag{5.9}$$

On the other hand,

$$\Delta^{\alpha-1}u(0) = u(1) - (\alpha-1)u(0). \tag{5.10}$$

And, by Lemma 2.2, part (ii) and (iii),

$$\Delta^2 k^{2-\beta}(n+1) = \frac{\beta(\beta-1)}{(n+2)(n+3)} k^{2-\beta}(n+1), \tag{5.11}$$

as well as

$$\Delta^3 k^{2-\beta}(n+1) = -\beta(\beta-1)(1+\beta)\frac{k^{2-\beta}(n+1)}{(n+2)(n+3)(n+4)}. \tag{5.12}$$

Using (5.9), (5.10), (5.11) we obtain

$$\begin{aligned}
&\Delta^2 k^{2-\beta}(n+1)\Delta^{\alpha-1}u(0) + \Delta^3 k^{2-\beta}(n+1)u(0)\\
&= \frac{\beta(\beta-1)}{(n+2)(n+3)} k^{2-\beta}(n+1)[u(1) - (\alpha-1)u(0)] - \beta(\beta-1)(1+\beta)\frac{k^{2-\beta}(n+1)}{(n+2)(n+3)(n+4)}u(0)\\
&= \frac{\beta(\beta-1)}{(n+2)(n+3)} k^{2-\beta}(n+1)[u(1) - (\alpha-1)u(0) - (1+\beta)\frac{1}{n+4}u(0)]\\
&= \frac{\beta(\beta-1)}{(n+2)(n+3)} k^{2-\beta}(n+1)\Big[u(1) - [(\alpha-1) + (1+\beta)\frac{1}{n+4}]u(0)\Big] \geq 0.
\end{aligned}$$

Then, by hypothesis (i) and the above inequality, we obtain from (5.8) that $\Delta^{\alpha+\beta}u(n+1) \geq 0$. Therefore the conclusions follows from hypothesis and Theorem 5.1.

$\mathcal{C}_4$: Suppose $(\alpha,\beta) \in \mathcal{C}_4$. If $\alpha+\beta = 2$, by Remark 3.4 with $l=2$, $m=1$, we have the identity

$$\Delta^2 u(n+1) = \Delta^{2-\alpha} \circ \Delta^\alpha u(n) + \Delta k^{\alpha-1}(n+1)\Delta^{\alpha-1}u(0) + \Delta^2 k^{\alpha-1}(n+1)u(0).$$

By part (ii) of Lemma 2.2 we have $\Delta^2 k^{1-\beta}(n+1) \geq 0$ and thus, by hypothesis (i), we conclude that $\Delta^2 u(n+1) \geq 0$. By hypothesis (iv) and Remark 5.2, we have proved the claim.

In other cases, by Proposition 3.3 part (v) with $l=2$, $m=1$, we obtain

$$\Delta^{\alpha+\beta}u(n) = \Delta^\beta \circ \Delta^\alpha u(n) + \Delta^2 k^{1-\beta}(n+1)u(0) + \Delta k^{1-\beta}(n+1)\Delta^{\alpha-1}u(0). \tag{5.13}$$

By part (ii) of Lemma 2.2, we have $\Delta^2 k^{1-\beta}(n+1) \geq 0$. Since $0 < 1-\beta < 1$, we obtain

$$\Delta k^{1-\beta}(n+1) = -\beta\frac{k^{1-\beta}(n+1)}{(n+2)} \geq -\beta\frac{k^{1-\beta}(1)}{2}.$$

On the other had, by hypothesis (iv) and (v), we obtain

$$\Delta^{\alpha-1}u(0) = u(1) - (\alpha-1)u(0) \geq (\alpha+\beta)u(0) - (\alpha-1)u(0) = (\beta+1)u(0).$$

Thus,

$$\begin{aligned}\Delta^{\alpha+\beta}u(n) =&\Delta^{\beta}\circ\Delta^{\alpha}u(n)+\Delta^{2}k^{1-\beta}(n+1)u(0)+\Delta k^{1-\beta}(n+1)\Delta^{\alpha-1}u(0)\\ \geq&\Delta^{\beta}\circ\Delta^{\alpha}u(n)-\frac{\beta}{2}(1-\beta)\Delta^{\alpha-1}u(0).\end{aligned}$$

Thus, if $(\alpha,\beta)\in\mathcal{C}_4$, we obtain $\Delta^{\alpha+\beta}u(n)\geq 0$, for all $n\in\mathbb{N}_0$. Therefore all the conclusions follows from hypothesis and Theorem 5.1.

$\mathcal{C}_5$: Suppose $(\alpha,\beta)\in\mathcal{C}_5$. In case $\alpha=2$ and $0\leq\beta\leq 1$ by Proposition 3.3 part (iv), with $l=2$, $m=1$, we have

$$\Delta^{\beta+2}u(n)=\Delta^{\beta}\circ\Delta^{2}u(n)+\Delta k^{1-\beta}(n+1)\Delta u(0)+\Delta^{2}k^{1-\beta}(n+1)u(0).$$

Thus by hypothesis (i), (4.6), and since $\Delta^2 k^{2-\beta}(n+1)\geq 0$ we obtain $\Delta^{\beta+2}u(n)\geq 0$. Hence we can apply Theorem 5.1 for $\beta\in[1,2]$ and obtain the claim. In other cases, by Proposition 3.3 part (vi), with $l=2$, $m=1$, we have

$$\Delta^{\alpha+\beta}u(n+1)=\Delta^{\beta}\circ\Delta^{\alpha}u(n)+\Delta^{3}k^{1-\beta}(n+1)u(0)+\Delta^{2}k^{1-\beta}(n+1)\Delta^{\alpha-2}u(0)+\Delta k^{1-\beta}(n+1)\Delta^{\alpha-1}u(0). \tag{5.14}$$

Moreover, by part (ii) Lemma 2.2,

$$\Delta^{2}k^{1-\beta}(n+1)=\beta(1+\beta)\frac{k^{1-\beta}(n+1)}{(n+2)(n+3)}. \tag{5.15}$$

Since $0<\alpha-2<1$, we also have

$$\begin{aligned}\Delta^{\alpha-2}u(0)&=\Delta(k^{3-\alpha}*u)(0)=(k^{3-\alpha}*u)(1)-(k^{3-\alpha}*u)(0)=(3-\alpha)u(0)+u(1)-u(0)\\&=u(1)-(\alpha-2)u(0).\end{aligned} \tag{5.16}$$

Moreover, by part (iii) in Lemma 2.2,

$$\Delta^{3}k^{1-\beta}(n+1)=-\beta(1+\beta)(\beta+2)\frac{k^{1-\beta}(n+1)}{(n+2)(n+3)(n+4)}. \tag{5.17}$$

And by part (i) in Lemma 2.2, since that $0<1-\beta<1$, then

$$\Delta k^{1-\beta}(n+1)=-\beta\frac{k^{1-\beta}(n+1)}{n+2}\geq-\beta\frac{k^{1-\beta}(1)}{2}. \tag{5.18}$$

On the other hand, for each $n\in\mathbb{N}_0$, we have $(\beta+2)(n+4)\geq\beta+2$, and then

$$\alpha(n+4)+(\beta+2)(n+4)\geq(\beta+2)+\alpha(n+4),$$

as well as

$$(\alpha+\beta)(n+4)\geq\beta+2+(\alpha-2)(n+4).$$

Therefore

$$\alpha+\beta\geq\frac{\beta+2}{n+4}+\alpha-2.$$

Thus, since that $u(1)\geq(\alpha+\beta)u(0)$ and $u(0)\geq 0$ we conclude that

$$u(1)\geq[\frac{\beta+2}{n+4}+\alpha-2]u(0). \tag{5.19}$$

Therefore, using (5.19), (5.16), (5.15) and (5.17) we obtain

$$\begin{aligned}
&\Delta^3 k^{1-\beta}(n+1)u(0) + \Delta^2 k^{1-\beta}(n+1)\Delta^{\alpha-2}u(0)\\
=&-\beta(1+\beta)(\beta+2)\frac{k^{1-\beta}(n+1)}{(n+2)(n+3)(n+4)}u(0) + \beta(1+\beta)\frac{k^{1-\beta}(n+1)}{(n+2)(n+3)}[u(1)-(\alpha-2)u(0)]\\
=&\beta(1+\beta)\frac{k^{1-\beta}(n+1)}{(n+2)(n+3)}[u(1)-(\alpha-2)u(0)) - (\beta+2)\frac{1}{(n+4)}u(0)]\\
=&\beta(1+\beta)\frac{k^{1-\beta}(n+1)}{(n+2)(n+3)}[u(1)-[(\alpha-2)+(\beta+2)\frac{1}{(n+4)}]u(0)] \geq 0.
\end{aligned}$$

Thus, by hypothesis (i) and (5.18), we obtain from (5.14)

$$\Delta^{\alpha+\beta}u(n) \geq 0, \text{ for all } n \in \mathbb{N}_1. \tag{5.20}$$

Therefore all the conclusions follows from hypothesis and Theorem 5.1.

□

Remark 5.7. *Note that in the sector $\mathcal{C}_4$ we have $1 < \alpha < 2$ and an easy calculation shows that $\Delta^{\alpha-1}u(0) = u(1) - (\alpha-1)u(0)$. In case of the sector $\mathcal{C}_5$, we have $2 < \alpha < 3$, and the identity $\Delta^{\alpha-1}u(0) = u(2) - (\alpha - 1)u(1) + \frac{(\alpha-1)(\alpha-2)}{2}u(0)$ holds.*

Remark 5.8. *Note that for $2 \leq \alpha+\beta < 3$ we obtain, after a calculation*

$$u(3) - (\alpha+\beta)u(2) + \frac{(\alpha+\beta)(\alpha+\beta-1)}{2}u(1) - \frac{(\alpha+\beta)(\alpha+\beta-1)(\alpha+\beta-2)}{6}u(0) = (\Delta^{\alpha+\beta}u)(0),$$

and hence the hypothesis (ii) in the previous theorem can be rewritten only in terms of the positivity of $(\Delta^{\alpha+\beta}u)(0)$.

Remark 5.9. *The previous theorem, improves the condition (i) on $(\Delta^\beta \circ \Delta^\alpha u)(n)$ in Theorems 7.13 and 7.17 of [26]. We recall that the condition (i) in [26, Theorem 7.13] is $(\Delta^\beta \circ \Delta^\alpha u)(n) \geq \beta(1+\beta)(\beta-1)\frac{(2-\beta)}{24}u(0)$, whereas the condition (i) of [26, Theorem 7.17] is $(\Delta^\beta \circ \Delta^\alpha u)(n) \geq \beta(1+\beta)(2+\beta)\frac{(1-\beta)}{24}u(0) + \beta\frac{(1-\beta)}{2}\Delta^{\alpha-1}u(0)$. In our findings, the condition on the right hand side of (5.5) is less restrictive. Moreover, our new hypothesis allows the conditions on the right hand side of (5.5) coincide when the pair (α,β) varies in the borders of each sector. This property was not posed in the results of [26].*

Moreover, we have added the hypothesis $(\Delta^{\alpha+\beta}u)(0) \geq 0$, which allow us to improve the results in [26, Theorems 7.11, 7.13, 7.15, 7.17] ensuring not only convexity but also positivity and monotonicity of a sequence on the set $\mathbb{N}_0$. We also observe that for $(\alpha,\beta) \in \mathcal{C}_5$ we added the conditions (ii) and (iii), which were missing in [26, Theorem 7.17].

Remark 5.10. *Note that if $\beta = 2$ the condition on the right hand side of (5.5) in Theorem 5.6 coincides in the regions $\mathcal{C}_1$ and $\mathcal{C}_2$. If $\alpha = 1$ then the condition on the right hand side of (5.5) coincides in the regions $\mathcal{C}_2$ and $\mathcal{C}_3$. If $\beta = 1$ then the conditions coincides in the regions $\mathcal{C}_3$ and $\mathcal{C}_4$ and, finally, if $\alpha = 2$ then the conditions coincide in the regions $\mathcal{C}_4$ and $\mathcal{C}_5$.*

The following example shows that the condition (ii) in Theorem 5.6 is necessary for convexity in $\mathcal{C}_1$ and $\mathcal{C}_3$.

Example 5.11. *Define a sequence $u : \mathbb{N}_0 \to \mathbb{R}$ by $u(0) = u(1) = 0$ and $u(n) := 2 - 2^{1-n}, n \in \mathbb{N}_2$. Let $\frac{4+\sqrt{2}}{2} < \alpha+\beta < 3$. Then the following assertions hold:*

(i) $(\Delta^{\beta} \circ \Delta^{\alpha} u)(n) \geq \begin{cases} 0 & \text{if } (\alpha,\beta) \in \mathcal{C}_1. \\ 0 & \text{if } (\alpha,\beta) \in \mathcal{C}_3. \end{cases}$

(ii) $u(2) \geq (\alpha+\beta)u(1) - \frac{1}{2}(\alpha+\beta)(\alpha+\beta-1)u(0)$;

(iii) $u(1) \geq (\alpha+\beta)u(0)$

(iv) $u(0) \geq 0$

(v) *u is positive, increasing and concave on $\mathbb{N}_2$.*

Indeed, it is clear that u is positive, increasing and the items $(ii), (iii), (iv)$ and (v) are verified. Proceeding analogously to Example 5.4, using Proposition 3.1, part (ii), with $a := k^{3-(\alpha+\beta)}, b := u, l_1 = 2, l_2 = 1$ we obtain for any $n \in \mathbb{N}_2$:

$$\begin{aligned}
\Delta^{\alpha+\beta}u(n) &= \sum_{j=1}^{n} \Delta^2 k^{3-(\alpha+\beta)}(j)\Delta u(n-j) + u(n+3) - (\alpha+\beta)u(n+2) + \frac{(\alpha+\beta)((\alpha+\beta)-1)}{2}u(n+1) \\
&\quad - \frac{(\alpha+\beta-1)(\alpha+\beta-2)}{2}u(n) \\
&= \sum_{j=1}^{n} \Delta^2 k^{3-(\alpha+\beta)}(j)\Delta u(n-j) + \frac{2^{n+3}-1}{2^{n+2}} - (\alpha+\beta)\frac{2^{n+2}-1}{2^{n+1}} + \frac{(\alpha+\beta)(\alpha+\beta-1)}{2}\frac{2^{n+1}-1}{2^n} \\
&\quad - \frac{(\alpha+\beta-1)(\alpha+\beta-2)}{2}\frac{2^n-1}{2^{n-1}}.
\end{aligned}$$

Since $\Delta u(n) \geq 0$, by Lemma 2.2 part (ii), we have $\sum_{j=1}^{n} \Delta^2 k^{3-\alpha}(j)\Delta u(n-j) \geq 0$. Thus, since $\alpha+\beta \in [\frac{4+\sqrt{2}}{2}, 3)$, and from the previous identity, we obtain

$$\Delta^{\alpha+\beta}u(n) \geq \frac{2(\alpha+\beta)^2 - 8(\alpha+\beta) + 7}{2^{n+2}} \geq 0, \quad n \in \mathbb{N}_2.$$

Note that $\frac{10+\sqrt{10}}{6} < \frac{4+\sqrt{2}}{2} < (\alpha+\beta) < 3$. Therefore, we also have

$$\begin{aligned}
\Delta^{\alpha+\beta}u(1) &\geq \Delta^2 k^{3-(\alpha+\beta)}(1)\Delta u(0) + u(4) - (\alpha+\beta)u(3) + \frac{(\alpha+\beta)(\alpha+\beta-1)}{2}u(2) \\
&= \frac{1}{8}(6(\alpha+\beta)^2 - 20(\alpha+\beta) + 15) \geq 0.
\end{aligned}$$

We conclude that $\Delta^{\alpha+\beta}u(n+1) \geq 0$ on $\mathbb{N}_0$. On the other hand, by (5.6), (5.8) and taking into account that $u(0) = u(1) = 0$, we obtain

$$(\Delta^{\beta} \circ \Delta^{\alpha} u)(n) = \begin{cases} \Delta^{\alpha+\beta}u(n+1) & \text{if } (\alpha,\beta) \in \mathcal{C}_1, \\ \Delta^{\alpha+\beta}u(n+1) & \text{if } (\alpha,\beta) \in \mathcal{C}_3. \end{cases}$$

This proves (i). We now prove that u is concave on $\mathbb{N}_2$. Indeed, by definition we obtain

$$\Delta^2 u(n) = u(n+2) - 2u(n+1) + u(n) = \frac{2^{n+2}-1}{2^{n+1}} - 2\frac{2^{n+1}-1}{2^n} + \frac{2^n-1}{2^{n-1}} = -\frac{1}{2^{n+1}} \leq 0,$$

proving the claim. However, note that $u(3) = \frac{7}{4} < (\alpha+\beta)\frac{3}{2} = (\alpha+\beta)u(2)$. It follows that the condition (ii) in Theorem 5.6 does not hold.

The next example shows that the condition $u(2) \geq (\alpha+\beta)u(1) - \frac{(\alpha+\beta)(\alpha+\beta-1)}{2}u(0)$ in Theorem 5.6 is necessary for convexity in $\mathcal{C}_2$.

Example 5.12. *Define the sequence $u : \mathbb{N}_0 \to \mathbb{R}$ by $u(n) := \gamma - \frac{1}{\gamma^{n-1}}$ where $\gamma > 1$ is fixed. Let $\frac{3\gamma+2+\sqrt{\gamma^2+4\gamma-4}}{2\gamma} \leq \alpha + \beta < 3$. The following statements are true:*

- $(\Delta^\beta \circ \Delta^\alpha u)(n) \geq 0$, *if* $(\alpha, \beta) \in \mathcal{C}_2$.
- $u(3) \geq (\alpha+\beta)u(2) - \frac{1}{2}(\alpha+\beta)(\alpha+\beta-1)u(1) + \frac{1}{6}(\alpha+\beta)(\alpha+\beta-1)(\alpha+\beta-2)u(0)$
- $u(1) \geq (\alpha+\beta)u(0)$
- $u(0) \geq 0$
- u *is positive, monotone increasing and concave on* $\mathbb{N}_0$.

In fact, we first observe that $u(0) = 0$, and $u(1) = \gamma - 1 > 0$. Also, we have that u is positive and $\Delta u(n) = u(n+1) - u(n) = \frac{\gamma-1}{\gamma^n} \geq 0$, i.e., u is monotone increasing on $\mathbb{N}_0$. Now, by Example 5.4, u is concave on $\mathbb{N}_0$ and, replacing α by $\alpha+\beta$ in Example 5.4, we have $\Delta^{\alpha+\beta}u(n) \geq 0$. Consequently, by (5.7) we obtain

$$(\Delta^\beta \circ \Delta^\alpha u)(n) = \Delta^{\alpha+\beta}u(n) - \Delta^2 k^{2-\beta}(n+1)u(0) \geq 0.$$

Notice that also have $u(3) - (\alpha+\beta)u(2) + \frac{1}{2}(\alpha+\beta)(\alpha+\beta-1)u(1) - \frac{1}{6}(\alpha+\beta)(\alpha+\beta-1)(\alpha+\beta-2)u(0) \geq 0$. Therefore all the assertions are verified, proving the claim. On the other hand, we have $u(2) = \frac{(\gamma-1)(\gamma+1)}{\gamma} < 2(\gamma-1) < (\alpha+\beta)(\gamma-1) = (\alpha+\beta)u(1) - \frac{(\alpha+\beta)(\alpha+\beta-1)}{2}u(0)$. It follows that the condition (iii) in Theorem 5.6 is necessary in order to ensure convexity.

Now, in the following example we show that the condition (iii) in Theorem 5.6 is necessary for positivity, monotonicity and convexity in $\mathcal{C}_4$ and $\mathcal{C}_5$.

Example 5.13. *Define a sequence $u : \mathbb{N}_0 \to \mathbb{R}$ by $u(0) = u(1) = 0$, $u(2) = -1$ and $u(n) := k^\gamma(n)$, $n \in \mathbb{N}_3$, $\gamma > 11$. Let $2 < \alpha + \beta < 3$. Then the following assertions hold:*

(i) $(\Delta^\beta \circ \Delta^\alpha u)(n) \geq \begin{cases} 0 & \text{if } (\alpha,\beta) \in \mathcal{C}_4. \\ \frac{\beta}{2}(1-\beta)u(2) & \text{if } (\alpha,\beta) \in \mathcal{C}_5. \end{cases}$

(ii) $u(3) \geq (\alpha+\beta)u(2) - \frac{1}{2}(\alpha+\beta)(\alpha+\beta-1)u(1) + \frac{1}{6}(\alpha+\beta)(\alpha+\beta-1)(\alpha+\beta-2)u(0)$;

(iii) $u(1) \geq (\alpha+\beta)u(0)$

(iv) $u(0) \geq 0$

(v) u *is non positive, non increasing and non concave on* $\mathbb{N}_0$.

In fact, since $k^\gamma(n) \geq 0$, it is clear that $(ii), (iii)$ and (iv) are verified. Moreover, u is non-positive and non-increasing because $u(2) = -1$. This property also implies $\Delta^2 u(0) = -1 < 0$ and therefore u cannot be convex on $\mathbb{N}_0$.

We check assertion (i). By Definition 2.4 and the semigroup property of the kernel k^γ (see (2.1)) we obtain, by Definition and part (iii) of Lemma 2.2, the following identities:

$$\begin{aligned}\Delta^{\alpha+\beta}u(n) &= \Delta^{\alpha+\beta}k^\gamma(n) = \Delta^3(k^{3-\alpha-\beta} * k^\gamma)(n) = \Delta^3(k^{3-\alpha-\beta+\gamma})(n) \\ &= (\gamma-(\alpha+\beta))(1+\gamma-(\alpha+\beta))(2+\gamma-(\alpha+\beta))\frac{k^{3-\alpha-\beta-\gamma}}{(n+1)(n+2)(n+3)},\end{aligned}$$

for all $n \in \mathbb{N}_3$. Therefore $\Delta^{\alpha+\beta}u(n) \geq 0$, $n \in \mathbb{N}_3$.

We now prove that $\Delta^{\alpha+\beta}u(n) \geq 0$ for $n = 0, 1, 2$. In fact, since $2 < \alpha+\beta < 3$, from Definition 2.4, we obtain $\Delta^{\alpha+\beta}u(0) = u(3) - (\alpha+\beta)u(2) = k^\gamma(3) + (\alpha+\beta) \geq 0$.

On the other hand, since $2<\alpha+\beta<3$ *and* $\gamma>11$*, we have* $\frac{1}{2}(\alpha+\beta)(\alpha+\beta-1)<3$ *and* $\gamma+3-4(\alpha+\beta)>1$. *Thus* $\frac{1}{4!}[\gamma+3-4(\alpha+\beta)](\gamma+2)(\gamma+1)\gamma>3$ *and we obtain*

$$\begin{aligned}\Delta^{\alpha+\beta}u(1)&=u(4)-(\alpha+\beta)u(3)+\frac{1}{2}(\alpha+\beta)(\alpha+\beta-1)u(2)\\&=k^{\gamma}(4)-(\alpha+\beta)k^{\gamma}(3)-\frac{1}{2}(\alpha+\beta)(\alpha+\beta-1)\\&=\frac{1}{4!}[\gamma+3-4(\alpha+\beta)](\gamma+2)(\gamma+1)\gamma-\frac{1}{2}(\alpha+\beta)(\alpha+\beta-1)\geq 0.\end{aligned}$$

Finally, since $2<\alpha+\beta<3$ *and* $\gamma>11$*, we have* $(\gamma+4)-5(\alpha+\beta)>0$. *Therefore,*

$$\begin{aligned}\Delta^{\alpha+\beta}u(2)&=u(5)-(\alpha+\beta)u(4)+\frac{1}{2}(\alpha+\beta)(\alpha+\beta-1)u(3)-\frac{1}{6}(\alpha+\beta)(\alpha+\beta-1)(\alpha+\beta-2)u(2)\\&=k^{\gamma}(5)-(\alpha+\beta)k^{\gamma}(4)+\frac{1}{2}(\alpha+\beta)(\alpha+\beta-1))k^{\gamma}(3)+\frac{1}{6}(\alpha+\beta)(\alpha+\beta-1)(\alpha+\beta-2)\\&\geq\frac{1}{5!}[(\gamma+3)(\gamma+4)-5(\alpha+\beta)(\gamma+3)+10(\alpha+\beta)(\alpha+\beta-1)](\gamma+2)(\gamma+1)\gamma\\&\geq\frac{1}{5!}[(\gamma+3)[(\gamma+4)-5(\alpha+\beta)]+10(\alpha+\beta)(\alpha+\beta-1)](\gamma+2)(\gamma+1)\gamma\geq 0.\end{aligned}$$

We conclude that $\Delta^{\alpha+\beta}u(n)\geq 0$ *for all* $n\in\mathbb{N}_0$*, as claimed. Moreover, since* $u(0)=u(1)=0$ *and* $u(2)=-1$*, by part* (v) *in Proposition 3.3, with* $l=2$, $m=1$ *we have in the sector* $\mathcal{C}_4$:

$$(\Delta^{\beta}\circ\Delta^{\alpha}u)(n)=\Delta^{\alpha+\beta}u(n)\geq 0,\quad n\in\mathbb{N}_0,$$

and by part (vi) *in Proposition 3.3, for* $l=3$, $m=1$ *we obtain for the sector* $\mathcal{C}_5$:

$$(\Delta^{\beta}\circ\Delta^{\alpha}u)(n)=\Delta^{\alpha+\beta}u(n)-\Delta k^{1-\beta}(n+1)u(2)\geq\frac{\beta}{2}(1-\beta)u(2),\quad n\in\mathbb{N}_0,$$

where we have used the inequality $-\Delta k^{1-\beta}(n+1)u(2)\geq\frac{\beta}{2}(1-\beta)u(2)$ *that follows from* (4.6)*. This proves* (i)*. However, note that* $u(2)=-1<0=(\alpha+\beta)u(1)$*. It follows that the condition* (iii) *in Theorem 5.6 does not hold.*

6. Proofs of Theorems 1.3 to 1.8

We devote this section to the proofs of Theorems 1.3, 1.4 and 1.5 concerning positivity, monotonicity and convexity in the interval $0\leq\nu<3$ and some of their properties, as well as Theorems 1.6, 1.7 and 1.8 regarding the same geometrical properties for the composition of two fractional difference operators as exhibes Figure 1. The research carried out in the previous Theorems 4.6, 4.16, 5.1 and 5.6 - as well as the transference principle - will be the key to prove the results.

Proof of Theorem 1.3 : In case $\nu=0$, by hypothesis (i), we have $\Delta_a^0 v(t)=v(t)\geq 0$ on $\mathbb{N}_{a+1}$. Moreover, $v(a)\geq 0$ implies that v is positive. For $0<\nu\leq 1$, the proof follows from [26, Corollary 5.6] and the transference principle (Theorem 2.6).

Proof of Theorem 1.4 : For $\nu=1$, we obtain $\Delta_a v(t)=v(t+1)-v(t)\geq 0$, i.e., v is monotone increasing and positive, using hypothesis (ii) and (iii). For $\nu\in(1,2]$, the conclusion follows from [26, Corollary 6.9] and using the transference principle.

Proof of Theorem 1.5 : In case $\nu=2$ the conclusion is clear from the hypothesis. Define $u:=\tau_a v$. Using Theorem 2.6 we have

$$\Delta^{\nu}u(n)=\tau_{a+3-\nu}\circ\Delta_a^{\nu}\circ\tau_{-a}u(n)=\tau_{a+3-\nu}\circ\Delta_a^{\nu}\circ\tau_a v(n)=\Delta_a^{\nu}v(t)\geq 0,$$

for $t := n + a + 3 - \nu \in \mathbb{N}_{a+3-\nu}$. The conclusion follows from the transference principle.

Proof of Theorem 1.6 : Follows from application of the transference principle and Theorem 4.6.

Proof of Theorem 1.7 : For $(\mu, \nu) = (0, 1) \in \mathcal{M}_1$ we have $(\Delta^1_{a+1} \circ \Delta^0_a v)(t) = \Delta_{a+1} v(t) \geq 0$ and by hypothesis we arrive at the conclusion. For $(\mu, \nu) = (0, 1) \in \mathcal{M}_2$ the reasoning is analogous. For $(\mu, \nu) = (1, 0) \in \mathcal{M}_2$ we have $(\Delta^0_a \circ \Delta^1_a v)(t) = \Delta_a v(t) \geq 0$ and hence v is positive and monotone increasing. For $(\mu, \nu) = (1, 0) \in \mathcal{M}_3$ we have $(\Delta^0_{a+1} \circ \Delta^1_a v)(t) = \Delta_{a+1} v(t) \geq 0$ and hence by hypothesis we obtain the conclusion. In other cases, the proof follows from the transference principle and Theorem 4.16.

Proof of Theorem 1.8 : Note that, when $(\mu, \nu) = (0, 2)$, $(\mu, \nu) = (1, 1)$ and $(\mu, \nu) = (2, 0)$ the result is immediate in the respective regions. Define $u := \tau_a v$. For $(\mu, \nu) \in \mathcal{C}_1$, using the transference principle we have,

$$\begin{aligned}
\Delta^\nu \circ \Delta^\mu u(n) =& \tau_{a+3-\nu} \circ \Delta^\nu_a \circ \tau_{-a} \circ \Delta^\mu u(n) = \tau_{a+3-\nu} \circ \Delta^\nu_a \circ \tau_{-a} \circ \tau_{a+1-\mu} \circ \Delta^\mu_a \circ \tau_{-a} u(n) \\
=& \tau_{a+3-\nu} \circ \Delta^\nu_a \circ \tau_{1-\mu} \circ \Delta^\mu_a \circ \tau_{-a} u(n) = \tau_{a+3-\nu} \circ \tau_{1-\mu} \circ \Delta^\nu_{a+1-\mu} \circ \Delta^\mu_a \circ \tau_{-a} u(n) \\
=& \tau_{a+4-\mu-\nu} \circ \Delta^\nu_{a+1-\mu} \circ \Delta^\mu_a v(n),
\end{aligned}$$

for each $n \in \mathbb{N}_0$. Therefore,

$$\Delta^\nu \circ \Delta^\mu u(n) = \Delta^\nu_{a+1-\mu} \circ \Delta^\mu_a v(t), \tag{6.1}$$

where $t := n + a + 4 - \mu - \nu \in \mathbb{N}_{a+4-\mu-\nu}$. For $(\mu, \nu) \in \mathcal{C}_2$, using again the transference principle, we obtain

$$\begin{aligned}
\Delta^\nu \circ \Delta^\mu u(n) =& \tau_{a+2-\nu} \circ \Delta^\nu_a \circ \tau_{-a} \circ \Delta^\mu u(n) = \tau_{a+2-\nu} \circ \Delta^\nu_a \circ \tau_{-a} \circ \tau_{a+1-\mu} \circ \Delta^\mu_a \circ \tau_{-a} u(n) \\
=& \tau_{a+2-\nu} \circ \Delta^\nu_a \circ \tau_{1-\mu} \circ \Delta^\mu_a \circ \tau_{-a} u(n) = \tau_{a+2-\nu} \circ \tau_{1-\mu} \circ \Delta^\nu_{a+1-\mu} \circ \Delta^\mu_a \circ \tau_{-a} u(n) \\
=& \tau_{a+3-\mu-\nu} \circ \Delta^\nu_{a+1-\mu} \circ \Delta^\mu_a v(n),
\end{aligned}$$

for each $n \in \mathbb{N}_0$. Therefore, we conclude that

$$\Delta^\nu \circ \Delta^\mu u(n) = \Delta^\nu_{a+1-\mu} \circ \Delta^\mu_a v(t), \tag{6.2}$$

where $t := n + a + 3 - \mu - \nu \in \mathbb{N}_{a+3-\mu-\nu}$. For $(\mu, \nu) \in \mathcal{C}_3$ we have for each $n \in \mathbb{N}_0$:

$$\begin{aligned}
\Delta^\nu \circ \Delta^\mu u(n) =& \tau_{a+2-\nu} \circ \Delta^\nu_a \circ \tau_{-a} \circ \Delta^\mu u(n) = \tau_{a+2-\nu} \circ \Delta^\nu_a \circ \tau_{-a} \circ \tau_{a+2-\mu} \circ \Delta^\mu_a \circ \tau_{-a} u(n) \\
=& \tau_{a+2-\nu} \circ \Delta^\nu_a \circ \tau_{2-\mu} \circ \Delta^\mu_a \circ \tau_{-a} u(n) = \tau_{a+2-\nu} \circ \tau_{2-\mu} \circ \Delta^\nu_{a+2-\mu} \circ \Delta^\mu_a \circ \tau_{-a} u(n) \\
=& \tau_{a+4-\mu-\nu} \circ \Delta^\nu_{a+2-\mu} \circ \Delta^\mu_a v(n).
\end{aligned}$$

We conclude that

$$\Delta^\nu \circ \Delta^\mu u(n) = \Delta^\nu_{a+2-\mu} \circ \Delta^\mu_a v(t), \tag{6.3}$$

where $t := n + a + 4 - \mu - \nu \in \mathbb{N}_{a+4-\mu-\nu}$. Next, for $(\mu, \nu) \in \mathcal{C}_4$ we obtain for each $n \in \mathbb{N}_0$:

$$\begin{aligned}
\Delta^\nu \circ \Delta^\mu u(n) =& \tau_{a+1-\nu} \circ \Delta^\nu_a \circ \tau_{-a} \circ \Delta^\mu u(n) = \tau_{a+1-\nu} \circ \Delta^\nu_a \circ \tau_{-a} \circ \tau_{a+2-\mu} \circ \Delta^\mu_a \circ \tau_{-a} u(n) \\
=& \tau_{a+1-\nu} \circ \Delta^\nu_a \circ \tau_{2-\mu} \circ \Delta^\mu_a \circ \tau_{-a} u(n) = \tau_{a+1-\nu} \circ \tau_{2-\mu} \circ \Delta^\nu_{a+2-\mu} \circ \Delta^\mu_a \circ \tau_{-a} u(n) \\
=& \tau_{a+3-\mu-\nu} \circ \Delta^\nu_{a+2-\mu} \circ \Delta^\mu_a v(n).
\end{aligned}$$

Therefore,

$$\Delta^\nu \circ \Delta^\mu u(n) = \Delta^\nu_{a+2-\mu} \circ \Delta^\mu_a v(t), \tag{6.4}$$

where $t := n + a + 3 - \mu - \nu \in \mathbb{N}_{a+3-\mu-\nu}$. Finally, for $(\mu, \nu) \in \mathcal{C}_5$

$$\begin{aligned}\Delta^{\nu} \circ \Delta^{\mu} u(n) =& \tau_{a+1-\nu} \circ \Delta_a^{\nu} \circ \tau_{-a} \circ \Delta^{\mu} u(n) = \tau_{a+1-\nu} \circ \Delta_a^{\nu} \circ \tau_{-a} \circ \tau_{a+3-\mu} \circ \Delta_a^{\mu} \circ \tau_{-a} u(n)\\ =& \tau_{a+1-\nu} \circ \Delta_a^{\nu} \circ \tau_{3-\mu} \circ \Delta_a^{\mu} \circ \tau_{-a} u(n) = \tau_{a+1-\nu} \circ \tau_{3-\mu} \circ \Delta_{a+3-\mu}^{\nu} \circ \Delta_a^{\mu} \circ \tau_{-a} u(n)\\ =& \tau_{a+4-\mu-\nu} \circ \Delta_{a+3-\mu}^{\nu} \circ \Delta_a^{\mu} v(n),\end{aligned}$$

for each $n \in \mathbb{N}_0$. Therefore,

$$\Delta^{\nu} \circ \Delta^{\mu} u(n) = \Delta_{a+3-\mu}^{\nu} \circ \Delta_a^{\mu} v(t), \tag{6.5}$$

where $t := n + a + 4 - \mu - \nu \in \mathbb{N}_{a+4-\mu-\nu}$. Moreover, if $(\mu, \nu) \in \mathcal{C}_4$, we have $\Delta^{\mu-1} u(0) = \tau_{a+2-\mu} \circ \Delta_a^{\mu-1} \circ \tau_{-a} u(0) = \Delta_a^{\mu-1} v(a+2-\mu)$, and if $(\mu, \nu) \in \mathcal{C}_5$, then $\Delta^{\mu-1} u(0) = \tau_{a+3-\mu} \circ \Delta_a^{\mu-1} \circ \tau_{-a} u(0) = \Delta_a^{\mu-1} v(a+3-\mu)$. Thus the conclusion follows of (6.1)-(6.5), hypothesis (i), (ii), (iii), (iv), (v) and Theorem 5.6.

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