ALMOST AUTOMORPHIC MILD SOLUTIONS TO FRACTIONAL DIFFERENTIAL EQUATIONS

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Abstract. We introduce the concept of α-resolvent families to prove the existence of almost automorphic mild solutions to the differential equation $D_t^\alpha u(t) = Au(t)+t^n f(t), 1 \leq \alpha \leq 2,\ n \in \mathbb{Z}_+$ considered in a Banach space X, where $f : \mathbb{R} \to X$ is almost automorphic. We also prove the existence and uniqueness of an almost automorphic mild solution of the semilinear equation $D_t^\alpha u(t) = Au(t)+f(t,u(t)), 1 \leq \alpha \leq 2$ assuming $f(t,x)$ is almost automorphic in t for each $x \in X$, satisfies a global Lipschitz condition and takes values on X. Finally, we prove also the existence and uniqueness of an almost automorphic mild solution of the semilinear equation $D_t^\alpha u(t) = Au(t) + f(t,u(t),u'(t)), 1 \leq \alpha \leq 2$, under analogous conditions as in the previous case.

1. Introduction

We study in this paper the almost automorphicity of semilinear fractional differential equations of the form

$$D_t^\alpha u(t) = Au(t) + t^n f(t,u(t),u'(t)), \quad t \in \mathbb{R}, \quad n \in \mathbb{Z}_+, \quad 1 \leq \alpha \leq 2, \tag{1.1}$$

where $A : D(A) \subset X \to X$ is the infinitesimal generator of an α-resolvent family defined on a complex Banach space X and $f : \mathbb{R} \times X \times X \to X$ is an almost almost automorphic function satisfying a suitable Lipschitz condition. The fractional derivative is understood in the Riemann-Liouville's sense.

A continuous function $f : \mathbb{R} \to X$ is said to be almost automorphic if for every sequence of real numbers $(s'_n)_{n\in\mathbb{N}}$ there exists a subsequence $(s_n)_{n\in\mathbb{N}} \subset (s'_n)_{n\in\mathbb{N}}$ such that

$$g(t) := \lim_{n\to\infty} f(t+s_n)$$

is well defined for each $t \in \mathbb{R}$, and

$$f(t) = \lim_{n\to\infty} g(t-s_n), \quad \text{for each } t \in \mathbb{R}.$$

Almost automorphicity is a generalization of the classical concept of an almost periodic function. It was introduced in the literature by S. Bochner and recently studied by several authors, including [2, 4, 6, 9, 17, 12] among others. A complete description of their properties and further applications to evolution equations can be found in the monographs [13] and [14] by G. M. N'Guérékata.

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Sufficient conditions for the existence of almost automorphic mild solutions of the semilinear evolution equation

$$u'(t) = Au(t) + f(t, u(t)), \quad t \in \mathbb{R}, \tag{1.2}$$

where f is almost automorphic has been studied in several papers in recent years. In [12], A is supposed to be the generator of an exponentially stable C_0-semigroup, and in [9] additionally f is supposed to be of the form $f(t, x) = P(t)Q(x)$. The existence and uniqueness of classical solutions to (1.2) was obtained in [7] under some additional conditions on A and f. The corresponding second order problem was recently studied in [17] supposing that A is the generator of an holomorphic semigroup.

The reason for studying equation (1.1) is that it appear in mathematical models of viscoelasticity [23] and other fields of science [16], [22]. In fact, equation (1.1) is equivalent to solve an integral equation of convolution type (see [3], [5]). It is also of interest in itself when one explores the borderline between diffusion ($\alpha = 1$) and wave propagation ($\alpha = 2$). While the study of the almost automorphic solutions of (1.1) in the case $\alpha = 1$ and $\alpha = 2$ was well studied in [7], [9], [12] and [17], to the knowledge of the authors no results yet exist for the fractional differential equations considered in this paper.

Our plan is as follows: In section 2, we introduce some preliminaries on fractional derivatives and the Mittag-Leffler function. We prove a representation formula in an special case, which give the explicit solution of the scalar fractional differential equation $D_t^\alpha u(t) = -\rho^\alpha u(t)$, with initial conditions expressed in terms of Riemann-Liouville fractional derivatives. Also we introduce the concept of α-resolvent families, which give us the necessary framework to apply an operator theoretical approach in the analysis of almost automorphic mild solutions for the abstract equation (1.1). In section 3, we treat the linear case

$$D_t^\alpha u(t) = Au(t) + t^n f(t), \quad 1 \le \alpha \le 2, \; n \in \mathbb{Z}_+$$

and prove our first main result (Theorem 3.4) which extends recent developments due to N'Guérékata [12], Basit and Pryde [2] and Bugajewski and Diagana [4]. Section 4 is devoted to the semilinear equation

$$D_t^\alpha u(t) = Au(t) + f(t, u(t)), \quad 1 \le \alpha \le 2. \tag{1.3}$$

We prove existence and uniqueness of a unique almost authomorphic mild solution under the assumption f is almost authomorphic and that some Lipchitz condition on f is satisfied (Theorem 4.1). It is remarkable that when $A = -\rho^\alpha$ ($\rho > 0$) and in contrast with the case $\alpha = 1$, our obtained condition turns to be better for some intermediate fractional derivative $\alpha \in (0, 1)$. Again our results extends [2], [4] and [12], where the case $\alpha = 1$ was considered. Finally, in section 5 we prove, under some conditions on the nonlinear term, existence and uniqueness of almost authomorphic mild solutions for the semilinear equation (Theorem 4.5):

$$D_t^\alpha u(t) = Au(t) + f(t, u(t), u'(t)), \quad 1 \le \alpha \le 2.$$

The scalar version is also studied, showing a different behavior as in the case of the scalar version of equation (1.3).

2. Preliminaries

Let $\alpha > 0$ and assume $u : [0, \infty) \to X$, where X is a complex Banach space. The Riemann-Liouville fractional derivative of u of order α is defined by

$$D_t^\alpha u(t) := \frac{d^m}{dt^m} \int_0^t g_{m-\alpha}(t-s)u(s)ds, \quad t > 0,$$

where

$$g_\beta(t) := \frac{t^{\beta-1}}{\Gamma(\beta)}, \quad t > 0, \quad \beta \geq 0.$$

When $\alpha = n$ is integer, we set $D_t^n := \frac{d^n}{dt^n}, n = 1, 2, ...$ The Laplace transform of a function $f \in L^1(\mathbb{R}_+, X)$ is defined by

$$\mathcal{L}(f)(\lambda) := \hat{f}(\lambda) := \int_0^\infty e^{-\lambda t} f(t)dt, \quad Re\lambda > \omega,$$

if the integral is absolutely convergent for $Re\lambda > \omega$. We have

$$\widehat{D_t^\alpha f}(\lambda) = \lambda^\alpha \hat{f}(\lambda) - \sum_{k=0}^{m-1} (g_{m-\alpha} * f)^{(k)}(0)\lambda^{m-1-k}. \tag{2.1}$$

The power function λ^α is uniquely defined as $\lambda^\alpha = |\lambda|^\alpha e^{iarg\lambda}$, with $-\pi < arg\lambda < \pi$. The Mittag-Leffler function (see e.g. [11]) is defined as follows:

$$E_{\alpha,\beta}(z) := \sum_{n=0}^\infty \frac{z^n}{\Gamma(\alpha n + \beta)} = \frac{1}{2\pi i} \int_{Ha} e^\mu \frac{\mu^{\alpha-\beta}}{\mu^\alpha - z} d\mu, \quad \alpha, \beta > 0, \quad z \in \mathbb{C},$$

where Ha is a Hankel path, i.e. a contour which starts and ends at $-\infty$ and encircles the disc $|\mu| \leq |z|^{1/\alpha}$ counter-clockwise. It is an entire function which provides a generalization of several usual functions, for example:

i) Exponential function: $E_{1,1}(z) = e^z$;
ii) Cosine functions: $E_{2,1}(z^2) = \cosh(z)$ and $E_{2,1}(-z^2) = \cos(z)$;
iii) Sine functions: $zE_{2,2}(z^2) = \sinh(z)$ and $zE_{2,2}(-z^2) = \sin(z)$.

The Laplace transform of the Mittag-Leffler function is given as (cf. [10, (A.27) p.267]):

$$\mathcal{L}(t^{\beta-1}E_{\alpha,\beta}(-\rho^\alpha t^\alpha))(\lambda) = \frac{\lambda^{\alpha-\beta}}{\lambda^\alpha + \rho^\alpha}, \quad Re\lambda > \rho^{1/\alpha}, \quad \rho > 0. \tag{2.2}$$

Let us consider the ordinary fractional differential equation

$$D_t^\alpha u(t) = -\rho^\alpha u(t), \quad 0 < \alpha < 2, \ \rho > 0. \tag{2.3}$$

According to the cases $0 < \alpha < 1$ and $1 < \alpha < 2$ equation (2.3) is referred as the fractional relaxation or the fractional oscillation equation, respectively cf. [10] and [11]. In the former case, it must be equipped with a single initial condition, say $(g_{1-\alpha} * u)(0) = u_0$, and in the latter with two initial conditions, say $(g_{2-\alpha} * u)(0) = u_0$ and $(g_{2-\alpha} * u)'(0) = u_1$. We note that on a series of examples from the field of viscoelasticity, Heymans and Podlubny [15] have demonstrated that it is possible to attribute physical meaning to initial conditions expressed in terms of Riemann-Liouville fractional derivatives, and that it is possible to obtain initial values for such initial conditions by appropriate measurements or observations.

The solution of (2.3) can be obtained by applying the Laplace transform (2.1) which together with (2.2) implies:

$$u(t) = t^{\alpha-1}E_{\alpha,\alpha}(-\rho^\alpha t^\alpha)u_0, \quad \alpha \in (0,1);$$

$$u(t) = t^{\alpha-2}E_{\alpha,\alpha-1}(-\rho^\alpha t^\alpha)u_0 + t^{\alpha-1}E_{\alpha,\alpha}(-\rho^\alpha t^\alpha)u_1, \quad \alpha \in (1,2).$$

In what follows, we will need an explicit description of the function

$$s_\alpha(t) = t^{\alpha-1}E_{\alpha,\alpha}(-\rho^\alpha t^\alpha), \quad \alpha \in (1,2)$$

whose Laplace transform is

$$\hat{s}_\alpha(\lambda) = \frac{1}{\lambda^\alpha + \rho^\alpha}, \quad Re\lambda > \rho^{1/\alpha}, \quad \rho > 0. \tag{2.4}$$

Proposition 2.1. *Let $1 < \alpha < 2$ and $\rho > 0$. For all $t \geq 0$ we have:*

$$\begin{aligned} s_\alpha(t) &= \frac{1}{\pi}\sin\pi\alpha \int_0^\infty e^{-rt}\frac{r^\alpha}{r^{2\alpha} + 2r^\alpha\rho^\alpha\cos\pi\alpha + \rho^{2\alpha}}dr \\ &- \frac{2}{\alpha\rho^{\alpha-1}}e^{t\rho\cos\pi/\alpha}\cos[t\rho\sin\pi/\alpha + \pi/\alpha]. \end{aligned} \tag{2.5}$$

Proof. In case $\rho = 1$ the proof is essentially contained in [10, p.244-247]. We sketch here the main steps in the general case. We first note the identity

$$\frac{1}{\lambda^\alpha + \rho^\alpha} = \frac{-1}{\rho^\alpha}(\lambda\frac{\lambda^{\alpha-1}}{\lambda^\alpha + \rho^\alpha} - 1). \tag{2.6}$$

Denote $e_\alpha(t) := E_{\alpha,1}(-\rho^\alpha t^\alpha)$. Then from (2.2) we get $\hat{e}_\alpha(\lambda) = \dfrac{\lambda^{\alpha-1}}{\lambda^\alpha + \rho^\alpha}$ for $Re\lambda > \rho^{1/\alpha}$. Hence from (2.4) and (2.6) we obtain

$$s_\alpha(t) = \frac{-1}{\rho^\alpha}e'_\alpha(t). \tag{2.7}$$

From the inversion complex formula for the Laplace transform, we have

$$e_\alpha(t) = \frac{1}{2\pi i}\int_{B_r} e^{\lambda t}\frac{\lambda^{\alpha-1}}{\lambda^\alpha + \rho^\alpha}d\lambda$$

where B_r denotes the Bromwich path, i.e. a line $Re(\lambda) = \sigma \geq \rho^{1/\alpha}$ and $Im(\lambda)$ running from $-\infty$ to $+\infty$.

In order to obtain a decomposition of e_α in two parts, we bend the Bromwich path of integration B_r into the equivalent Hankel path $Ha(\rho^{1/\alpha})$, a loop which starts from $-\infty$ along the lower side of the negative real axis, encircles the circular disc $|\lambda| = \rho^{1/\alpha}$ in the positive sense and ends at $-\infty$ along the upper side of the negative real axis. One obtains

$$e_\alpha(t) = f_\alpha(t) + g_\alpha(t), \tag{2.8}$$

with

$$f_\alpha(t) = \frac{1}{2\pi i}\int_{Ha(\epsilon)} e^{\lambda t}\frac{\lambda^{\alpha-1}}{\lambda^\alpha + \rho^\alpha}d\lambda$$

where now the Hankel path $Ha(\epsilon)$ denotes a loop constituted by a small circle $|\lambda| = \epsilon$ with $\epsilon \to 0$ and by the two borders of the cut negative real axis, and

$$g_\alpha(t) = e^{s_0 t} Res(\frac{\lambda^{\alpha-1}}{\lambda^\alpha + \rho^\alpha})|_{s_0} + e^{s_1 t} Res(\frac{\lambda^{\alpha-1}}{\lambda^\alpha + \rho^\alpha})|_{s_1}$$

where $s_0 = \rho e^{i\pi/\alpha}$ and $s_1 = \rho e^{-i\pi/\alpha}$ are the poles of $\frac{\lambda^{\alpha-1}}{\lambda^\alpha+\rho^\alpha}$, $(1 < \alpha < 2)$. Note that s_0 and s_1 are located in the left half plane. Then one obtains

$$g_\alpha(t) = \frac{2}{\alpha} e^{\rho t \cos(\pi/\alpha)} \cos[\rho t \sin(\pi/\alpha)];\ 1 < \alpha < 2. \tag{2.9}$$

On the other hand, the contribution from the Hankel path $Ha(\epsilon)$ as $\epsilon \to 0$ is provided by

$$f_\alpha(t) = \int_0^\infty e^{-rt} K_\alpha(r) dr, \tag{2.10}$$

with

$$\begin{aligned} K_\alpha(r) &= -\frac{1}{\pi} Im\{\frac{\lambda^{\alpha-1}}{\lambda^\alpha + \rho^\alpha}|_{\lambda = re^{i\pi}}\} \\ &= \frac{1}{\pi} \frac{\rho^\alpha r^{\alpha-1} \sin \alpha\pi}{r^{2\alpha} + 2r^\alpha \rho^\alpha \cos(\alpha\pi) + \rho^{2\alpha}}. \end{aligned} \tag{2.11}$$

Hence

$$e'_\alpha(t) = -\int_0^\infty r e^{-rt} K_\alpha(r) dr + \frac{2\rho}{\alpha} e^{\rho t \cos(\pi/\alpha)} \cos[\rho t \sin(\pi/\alpha) + \pi/\alpha], \tag{2.12}$$

and therefore from (2.12) and (2.7) we obtain (2.5).

□

Remark 2.2. We note that in order to satisfy $e_\alpha(0) = 1$, we find from (2.8) and (2.9)

$$1 = f_\alpha(0) + g_\alpha(0) = f_\alpha(0) + \frac{2}{\alpha}.$$

Hence from (2.10) and (2.11),

$$\frac{1}{\pi} \int_0^\infty \frac{r^{\alpha-1} \sin \alpha\pi}{r^{2\alpha} + 2r^\alpha \rho^\alpha \cos(\alpha\pi) + \rho^{2\alpha}} dr = \frac{1}{\rho^\alpha}(1 - \frac{2}{\alpha}).$$

In order to give an operator theoretical approach to equation (1.1) we introduce the following definition.

Definition 2.3. Let A be a closed and linear operator with domain $D(A)$ defined on a Banach space X and $\alpha > 0$. We call A the generator of an α-resolvent family if there exists $\omega \geq 0$ and a strongly continuous function $S_\alpha : \mathbb{R}_+ \to \mathcal{B}(X)$ such that $\{\lambda^\alpha : Re\lambda > \omega\} \subset \rho(A)$ and

$$(\lambda^\alpha - A)^{-1} x = \int_0^\infty e^{-\lambda t} S_\alpha(t) x dt, \quad Re\lambda > \omega, \quad x \in X.$$

In this case, $S_\alpha(t)$ is called the α-resolvent family generated by A.

Because of the uniqueness of the Laplace transform, a 1-resolvent family is the same as a C_0-semigroup whereas that a 2-resolvent family corresponds to the concept of sine family, see [1, Section 3.15].

We note that α-resolvent families are a particular case of (a,k)-regularized families introduced in [18]. These are studied in a series of several papers in recent years (see [19], [20], [21], [25]). According to [18] an α-resolvent family $S_\alpha(t)$ corresponds to a (g_α, g_α)-regularized family.

Notably, α-resolvent families are also present in [3, p.62] (see formula (4.33)) where some properties are studied in the context of vector-valued $L^p(\mathbb{R}, X)$ spaces.

As in the situation of C_0-semigroups we have diverse relations of an α-resolvent family and its generator. The following result is a direct consequence of [18, Proposition 3.1 and Lemma 2.2].

Proposition 2.4. *Let $1 \le \alpha \le 2$ and let $S_\alpha(t)$ be an α-resolvent family on X with generator A. Then the following holds:*

(a) $S_\alpha(t)D(A) \subset D(A)$ and $AS_\alpha(t)x = S_\alpha(t)Ax$ for all $x \in D(A), t \ge 0$;

(b) Let $x \in D(A)$ and $t \ge 0$. Then

$$S_\alpha(t)x = g_\alpha(t)x + \int_0^t g_\alpha(t-s)AS_\alpha(s)xds. \tag{2.13}$$

In particular $\frac{d}{dt}S_\alpha(t)x$ exist.

(c) Let $x \in X$ and $t \ge 0$. Then $\int_0^t g_\alpha(t-s)S_\alpha(s)xds \in D(A)$ and

$$S_\alpha(t)x = g_\alpha(t)x + A\int_0^t g_\alpha(t-s)S_\alpha(s)xds.$$

In particular, $S_\alpha(0) = g_\alpha(0)$.

Remark 2.5. Let $1 < \alpha < 2$. Taking Laplace transform to the equation

$$D_t^\alpha u(t) = Au(t), \quad (g_{2-\alpha} * u)(0) = 0, \quad (g_{2-\alpha} * u)'(0) = x \tag{2.14}$$

we obtain formally that the Laplace transform of their solution is $(\lambda^\alpha - A)^{-1}$. In consequence, equation (2.14) is well posed if and only if A is the generator of an α-resolvent family.

Let $1 \le \alpha \le 2$. If an operator A with domain $D(A)$ is the infinitesimal generator of an α-resolvent family $S_\alpha(t)$ then for all $x \in D(A)$ we have

$$Ax = \frac{\Gamma(2\alpha - 1)}{\Gamma(\alpha)} \lim_{t \to 0^+} \frac{\Gamma(\alpha)S_\alpha(t)x - t^{\alpha-1}x}{t^{2\alpha-1}},$$

see [21, Theorem 2.1]. For example, the limit case $S_1(t)$ corresponds to the generator of a C_0-semigroup and $S_2(t)$ actually corresponds to the generator of a sine family.

A characterization of generators of α-resolvent families, analogous to the Hille-Yosida Theorem for C_0 semigroups, can be directly deduced from [18, Theorem 3.4]. Results on perturbation, approximation, representation as well as ergodic type theorems can be also deduced from the more general context of (a,k) regularized resolvents (see [19], [20],

[21] and [25]). We finish this section with the following practical criterion for α-resolvent families.

Theorem 2.6. *Let A be the generator of a strongly continuous cosine family. Then A is the generator of an α-resolvent family for all $1 \leq \alpha < 2$.*

Proof. Since A generates a cosine family, then by the subordination principle of solution operators [3, Theorem 3.1] we have that A generates a strongly continuous family $R_\alpha(t)$, exponentially bounded, such that

$$\hat{R}_\alpha(\lambda) = \lambda^{\alpha-1}(\lambda^\alpha - A)^{-1},$$

for all λ sufficiently large. Define

$$S_\alpha(t)x = \frac{d}{dt}\int_0^t g_\alpha(t-s)R_\alpha(s)x ds, \quad t \geq 0, \quad x \in X.$$

Then it is clear that $S_\alpha(t)$ is strongly continuous and,

$$\hat{S}_\alpha(\lambda) = \lambda \widehat{g_\alpha * R_\alpha}(\lambda) = \lambda \frac{1}{\lambda^\alpha}\hat{R}_\alpha(\lambda) = (\lambda^\alpha - A)^{-1},$$

for all λ sufficiently large. Therefore, A generates an α-resolvent family. □

3. Almost Automorphic Mild Solutions

In this section we consider the existence and uniqueness of automorphic mild solutions to the fractional evolution semilinear equation

$$D_t^\alpha u(t) = Au(t) + t^n f(t, u(t), u'(t)), \quad t \in \mathbb{R} \tag{3.1}$$

where A is the generator of an α-resolvent family.

As a consequence of the definition of an almost automorphic function given in the introduction, the following properties hold (cf [14]) : let $f, g : \mathbb{R} \to X$ be almost automorphic functions and let $\lambda \in \mathbb{R}$, then $f + g, \lambda f$ and f_λ are almost automorphic, where $f_\lambda(t) := f(t+\lambda)$. Moreover, the range $R(f)$ of f is relatively compact, therefore it is bounded. Almost automorphic functions constitute a Banach space $AA(X)$ when it is endowed with the sup norm:

$$||f||_\infty := \sup_{t\in\mathbb{R}} ||f(t)||.$$

If $w_n(t) := (1+|t|)^n, n \in \mathbb{N}_0, t \in \mathbb{R}$ and $f : \mathbb{R} \to X$ we set

$$||f||_{w_n,\infty} := ||f/w_n||_\infty.$$

We consider the following weighted classes (see [2])

$$AA_{w_n}(X) := \{w_n f : f \in AA(X)\} \text{ and } C_{w_n,0}(\mathbb{R}, X) := \{w_n f : f \in C_0(\mathbb{R}; X)\}$$

Then one can check that $AA_{w_n}(X)$ and $C_{w_n,0}(\mathbb{R}, X)$ are Banach spaces endowed with the norm $|| \cdot ||_{w_n,\infty}$. Moreover, it was proved in [2] that $AA_{w_n}(X) + C_{w_n,0}(\mathbb{R}, X)$ is a closed subspace of $BC(\mathbb{R}, X)$ and that the sum is (topologically) direct, see [2, Theorem 1.6] and the remark before Definition 3.2 in the cited paper.

The following result on the almost automorphicity of the convolution is the key for the results of this paper. It can be directly deduced from [4, Theorem 2.1]. We give here a detailed proof for the sake of completeness.

Lemma 3.1. *Let $\{S(t)\}_{t\geq 0} \subset \mathcal{B}(X)$ be a strongly continuous family of bounded and linear operators such that*

$$\|S(t)\| \leq \phi(t) \text{ for all } t \in \mathbb{R}_+ \text{ with } \phi \in L^1(\mathbb{R}_+).$$

If $f : \mathbb{R} \to X$ is an almost automorphic function then

$$\int_{-\infty}^{t} S(t-s)f(s)\, ds \in AA(X).$$

Proof. Let $(s_n') \subset \mathbb{R}$ be an arbitrary sequence. Since $f \in AA(X)$ there exists a subsequence (s_n) of (s_n') such that

$$\lim_{n\to\infty} f(t+s_n) = g(t), \text{ for all } t \in \mathbb{R}$$

and

$$\lim_{n\to\infty} g(t-s_n) = f(t), \text{ for all } t \in \mathbb{R}.$$

We define $F(t) = \int_{-\infty}^{t} S(t-s)f(s)\, ds$ and $G(t) = \int_{-\infty}^{t} S(t-s)g(s)\, ds$. Now consider

$$\begin{aligned} F(t+s_n) &= \int_{-\infty}^{t+s_n} S(t+s_n-s)f(s)\, ds \\ &= \int_{-\infty}^{t} S(t-\sigma)f(\sigma+s_n)\, d\sigma. \end{aligned}$$

Note that

$$\|F(t+s_n)\| \leq \|\phi\|_1 \|f\|_\infty \text{ and } \|G(t)\| \leq \|\phi|_1 \|g\|_\infty$$

and by continuity of $S(\cdot)x$ we have $S(t-\sigma)f(\sigma+s_n) \to S(t-\sigma)g(\sigma)$, as $n \to \infty$ for each $\sigma \in \mathbb{R}$ fixed and any $t \geq \sigma$. Then by the Lebesgue's dominated convergence theorem,

$$F(t+s_n) \to G(t) \text{ as } n \to \infty, \text{ for all } t \in \mathbb{R}.$$

In similar way we can show that

$$G(t-s_n) \to F(t) \text{ as } n \to \infty, \text{ for all } t \in \mathbb{R}.$$

□

Let $1 < \alpha < 2$ and assume that A generates a bounded α-resolvent family $S_\alpha(t)$ on X, and that $f \in L^1_{loc}(\mathbb{R}_+, X)$ is given. Then the unique solution of the problem

$$D_t^\alpha u(t) = Au(t) + f(t), \quad (g_{2-\alpha} * u)(0) = 0, \quad (g_{2-\alpha} * u)'(0) = x$$

is given by

$$u(t) = S_\alpha(t)x + \int_0^t S_\alpha(t-s)f(s)ds, \quad t \geq 0.$$

For our purposes, it is not natural to specify an initial value, and we therefore extend the previous terminology as follows.

Definition 3.2. Let A be the generator of an α-resolvent family $S_\alpha(t)$. A function $u : \mathbb{R} \to X$ satisfying the equation

$$u(t) = \int_{-\infty}^{t} S_\alpha(t-s)s^n F(s,u(s),u'(s))ds, \text{ for all } t \in \mathbb{R} \tag{3.2}$$

is called a mild solution on $\mathbb{R}$ of the equation

$$D_t^\alpha u(t) = Au(t) + t^n F(t,u(t),u'(t)), \quad t \in \mathbb{R},\ n \in \mathbb{Z}_+.$$

Remark 3.3. We note that the above definition is the natural extension of the usual concept of mild solution in the boundary cases $\alpha = 1$ and $\alpha = 2$. In fact, in the first case $T(t) = S_1(t)$ is the C_0-semigroup generated by A and $u(t) = \int_{-\infty}^{t} T(t-s)f(s)ds$. (Here we denote $f(s) := s^n F(s,u(s),u'(s)))$. Then we have for all $t > a, a \in \mathbb{R}$, the identities

$$\begin{aligned}
T(t-a)u(a) \quad &+ \quad \int_a^t T(t-s)f(s)ds \\
&= \quad T(t-a)\int_{-\infty}^{a} T(a-s)f(s)ds + \int_a^t T(t-s)f(s)ds \\
&= \quad \int_{-\infty}^{a} T(t-s)f(s)ds + \int_a^t T(t-s)f(s)ds \\
&= \quad \int_{-\infty}^{t} T(t-s)f(s)ds = u(t).
\end{aligned}$$

In the second case we have that $S(t) := S_2(t)$ is the sine family generated by A and $u(t) = \int_{-\infty}^{t} S(t-s)f(s)ds$. Therefore, given $a \in \mathbb{R}$ and denoting by $C(t) := S'(t)$ the cosine function (cf. [1] or [8]), we have

$$\begin{aligned}
C(t-a)u(a) \quad &+ \quad S(t-a)u'(a) = C(t-a)\int_{-\infty}^{a} S(a-s)f(s)\,ds \\
&+ \quad S(t-a)\int_{-\infty}^{a} C(a-s)f(s)\,ds \\
&= \quad \int_{-\infty}^{a} [C(t-a)S(a-s) + S(t-a)C(a-s)]f(s)\,ds.
\end{aligned}$$

Since $S(t+s) = C(s)S(t) + S(s)C(t)$, we obtain

$$C(t-a)u(a) + S(t-a)u'(a) = \int_{-\infty}^{a} S(t-s)f(s)\,ds.$$

In consequence we get

$$u(t) = \int_{-\infty}^{t} S(t-s)f(s)\,ds = [C(t-a)u(a) + S(t-a)u'(a)] + \int_a^t S(t-s)f(s)\,ds,$$

which is the usually called mild solution for the second order abstract Cauchy problem.

Note that in case of $1 < \alpha < 2$ there is no analogue of the semigroup property $T(t+s) = T(t)T(s)$ or cosine functional equation $C(t+s) + C(t-s) = 2C(t)C(s)$ which play the crucial role in the developing of the corresponding theories. This is due to the nonlocal character of the fractional differentiation leading always to some presence of memory.

The following is the main result of this section. It corresponds to an extension of [12, Theorem 3.1].

Theorem 3.4. *Let $n \in \mathbb{Z}_+$. Assume that A generates an α-resolvent family $\{S_\alpha(t)\}_{t\geq 0}$ for some $1 \leq \alpha < 2$ satisfying*

$$\|s^k S_\alpha(t)\| \leq \phi_{\alpha,k}(t), \ t \geq 0, \ \textit{with } \phi_{\alpha,k} \in L^1(\mathbb{R}_+) \tag{3.3}$$

for all $k = 0, 1, ..., n$. Let $f \in AA(\mathbb{X})$. Then the equation

$$D_t^\alpha u(t) = Au(t) + t^n f(t), \quad t \in \mathbb{R},$$

has a mild solution $u \in AA_{w_n}(X) \oplus C_{w_n,0}(\mathbb{R}, X)$.

Proof. Let $u(t) = \displaystyle\int_0^\infty S_\alpha(s)(t-s)^n f(t-s)\, ds$ then

$$\begin{aligned} \int_0^\infty S_\alpha(s)(t-s)^n f(t-s)\, ds &= t^n \int_0^\infty S_\alpha(s) f(t-s)\, ds \\ &+ \sum_{k=1}^n (-1)^k \begin{pmatrix} n \\ k \end{pmatrix} t^{n-k} \int_0^\infty s^k S_\alpha(s) f(t-s)\, ds \\ &=: u_1(t) + u_2(t). \end{aligned}$$

(in case $n = 0$ we take $u_2(t) \equiv 0$). By Lemma 3.1, $u_1 \in AA_{w_n}(X)$. We will show that $u_2 \in C_{w_n,0}(\mathbb{R}, X)$. Indeed, by (3.3) we have $s^k S_\alpha(\cdot) \in L^1(\mathbb{R}_+, \mathcal{B}(X))$ for all $k = 0, 1, ..., n$. So

$$\left\| \int_0^\infty s^k S_\alpha(s) f(t-s)\, ds \right\| \leq \int_0^\infty \|s^k S_\alpha(s) f(t-s)\|\, ds \leq \|f\|_\infty ||\phi_k||_1$$

for all $k = 0, 1, ..., n$. Since $\displaystyle\lim_{|t|\to\infty} \frac{t^r}{(1+|t|)^n} = 0$, we have

$$t^r \int_0^\infty s^k S_\alpha(s) f(t-s)\, ds \in C_{w_n,0}(\mathbb{R}, \mathbb{X}), \quad 0 \leq r < n$$

and this shows $u_2 \in C_{w_n,0}(\mathbb{R}, X)$. □

Remark 3.5. Note that the case $\alpha = 2$ is not covered by the above theorem, even in the case $n = 0$. This is due to the fact that a sine family cannot be stable, as it was recently proved in [24, Theorem 2.3]. On the other hand, in case $\alpha = 1$ and $n = 0$, Theorem 3.4 was proved by N'Guérékata [12], and in case $\alpha = 1, n \in \mathbb{N}$ by Basit and Pryde [2].

Corollary 3.6. *Assume that A generates an α-resolvent family $\{S_\alpha(t)\}_{t\geq 0}$ such that*

$$\|S_\alpha(t)\| \leq \phi_\alpha(t), \ \textit{for all } t \geq 0, \ \textit{with } \phi_\alpha \in L^1(\mathbb{R}_+),$$

and let $f \in AA(X)$. Then for all $1 \leq \alpha < 2$, the equation

$$D_t^\alpha u(t) = Au(t) + f(t), \quad t \in \mathbb{R},$$

has a mild solution $u \in AA(X)$.

The following result is an extension of [4, Theorem 3.1] where the case $\alpha = 1$ is proved.

Corollary 3.7. *Let $f:\mathbb{R}\to\mathbb{R}$ be an almost automorphic function and let $\rho>0$ be a real number. Then for all $1<\alpha<2$, the equation*

$$D_t^\alpha u(t) = -\rho^\alpha u(t) + f(t), \quad t\in\mathbb{R}, \tag{3.4}$$

has a mild almost automorphic solution given by

$$u(t) = \int_{-\infty}^t S_\alpha(t-s)f(s)ds, \quad t\in\mathbb{R},$$

where

$$\begin{aligned} S_\alpha(t) &= \frac{1}{\pi}\sin\pi\alpha\int_0^\infty e^{-rt}\frac{r^\alpha}{r^{2\alpha}+2r^\alpha\rho^\alpha\cos\pi\alpha+\rho^{2\alpha}}dr \\ &- \frac{2}{\alpha\rho^{\alpha-1}}e^{t\rho\cos\pi/\alpha}\cos[t\rho\sin\pi/\alpha+\pi/\alpha], \quad t\geq 0. \end{aligned} \tag{3.5}$$

Proof. The proof that (3.5) is an α-resolvent family follows from Proposition 2.1. We will prove that

$$|S_\alpha(t)|\leq\varphi_\alpha(t), \quad t\geq 0,$$

where $\varphi_\alpha\in L^1(\mathbb{R}_+)$. In fact, note that $r^{2\alpha}+2r^\alpha\rho^\alpha\cos\pi\alpha+\rho^{2\alpha}=(r^\alpha\cos(\pi\alpha)+\rho^\alpha)^2+(r^\alpha\sin(\pi\alpha))^2\geq 0$, then we have

$$\begin{aligned} |S_\alpha(t)| &\leq \frac{1}{\pi}|\sin\pi\alpha|\int_0^\infty e^{-rt}\frac{r^\alpha}{r^{2\alpha}+2r^\alpha\rho^\alpha\cos\pi\alpha+\rho^{2\alpha}}dr \\ &+ \frac{2}{\alpha\rho^{\alpha-1}}e^{t\rho\cos\pi/\alpha} =: \varphi_\alpha(t). \end{aligned}$$

Applying Fubini's theorem and noting that $\cos\pi/\alpha<0$ and $\sin\pi\alpha<0$ for $1<\alpha<2$, we get

$$\begin{aligned} \int_0^\infty|\varphi_\alpha(t)|dt &= \frac{1}{\pi}|\sin\pi\alpha|\int_0^\infty\int_0^\infty e^{-rt}\frac{r^\alpha}{r^{2\alpha}+2r^\alpha\rho^\alpha\cos\pi\alpha+\rho^{2\alpha}}dtdr \\ &+ \frac{2}{\alpha\rho^{\alpha-1}}\int_0^\infty e^{t\rho\cos\pi/\alpha}dt \\ &= \frac{-\sin\pi\alpha}{\pi}\int_0^\infty\frac{r^{\alpha-1}}{r^{2\alpha}+2r^\alpha\rho^\alpha\cos\pi\alpha+\rho^{2\alpha}}dr \\ &+ \frac{2}{\alpha\rho^\alpha(-\cos(\pi/\alpha))} \end{aligned}$$

The first term is equal to $\frac{-1}{\rho^\alpha}(1-\frac{2}{\alpha})$ by Remark 2.2. Therefore $\varphi_\alpha\in L^1(\mathbb{R}_+)$ and

$$||\varphi_\alpha||_1 = \frac{2}{\alpha\rho^\alpha}-\frac{1}{\rho^\alpha}-\frac{2}{\alpha\rho^\alpha\cos(\pi/\alpha)} \tag{3.6}$$

□

Remark 3.8. Define $l(\alpha):=||\varphi_\alpha||_1^{-1}$. Then for $1\leq\alpha\leq 2$ we have $l(1)=\frac{\rho}{3}$, $l(2)=0$ and $l(\alpha)$ has a maximum at some point $\alpha_0(\rho)\in(1,2)$. We observe that the point $\alpha_0(\rho)$ goes to 1 as ρ goes to 0 and, conversely, the point $\alpha_0(\rho)$ goes to 2 as ρ goes to ∞.

4. Semilinear Fractional Differential Equations

In what follows we denote:

$$l(\alpha,\rho) = \left(\frac{2}{\alpha\rho^\alpha} - \frac{1}{\rho^\alpha} - \frac{2}{\alpha\rho^\alpha\cos(\pi/\alpha)}\right)^{-1}, \quad \rho > 0,$$

whenever $1 < \alpha < 2$. Our main result in this section extends [12, Theorem 3.2] to the fractional case.

Theorem 4.1. *Assume that A generates an α-resolvent family $\{S_\alpha(t)\}_{t\geq 0}$ such that*

$$\|S_\alpha(t)\| \leq \phi_\alpha(t), \text{ for all } t \geq 0, \text{ with } \phi_\alpha \in L^1(\mathbb{R}_+).$$

Let $f : \mathbb{R} \times X \to X$ almost automorphic in t for each $x \in X$ and satisfies a Lipschitz condition in x uniformly in t, that is,

$$\|f(t,x) - f(t,y)\| \leq L\|x-y\|, \text{ for all } x, y \in X.$$

Then

$$D_t^\alpha u(t) = Au(t) + f(t,u(t)), \quad t \in \mathbb{R}, \tag{4.1}$$

has a unique almost automorphic mild solution whenever $L < ||\phi_\alpha||_1^{-1}$.

Proof. We define the operator $F : AA(X) \mapsto AA(X)$ by

$$(F\varphi)(t) := \int_{-\infty}^t S_\alpha(t-s)f(s,\varphi(s))\,ds, \quad t \in \mathbb{R}.$$

In view of [14, Theorem 2.2.6] and Lemma 3.1, F is well defined. Then for $\varphi_1, \varphi_2 \in AA(X)$ we have:

$$\begin{aligned}
\|F\varphi_1 - F\varphi_2\|_\infty &= \sup_{t\in\mathbb{R}} \left\| \int_{-\infty}^t S_\alpha(t-s)[f(s,\varphi_1(s)) - f(s,\varphi_2(s))]ds \right\| \\
&\leq L \sup_{t\in\mathbb{R}} \int_0^\infty \|S_\alpha(\tau)\| \|\varphi_1(t-\tau) - \varphi_2(t-\tau)\| d\tau \\
&\leq L\|\varphi_1 - \varphi_2\|_\infty \int_0^\infty \phi_\alpha(\tau) d\tau.
\end{aligned}$$

This proves that F is a contraction, so by the Banach fixed point theorem there exists a unique $u \in AA(X)$, such that $Fu = u$, that is $u(t) = \displaystyle\int_{-\infty}^t S_\alpha(t-s)f(s,u(s))ds.$ □

Remark 4.2. We note that if f is a function of the form $f(t,x) = P(t)Q(x)$ and under appropriate conditions on P and Q, it was obtained in [9] the existence of almost automorphic mild solutions to (4.1) in case $\alpha = 1$, with f being not necessarily Lipschitzian.

The following corollary is an extension of [4, Theorem 3.2] where the case $\alpha = 1$ was proved.

Corollary 4.3. *Let $\rho > 0$ be a real number. Let $f : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ almost automorphic in the first variable and satisfies a Lipschitz condition in the second variable, that is,*

$$\|f(t,x) - f(t,y)\| \leq L\|x-y\|, \text{ for all } x, y \in \mathbb{R}.$$

Then

$$D_t^\alpha u(t) = -\rho^\alpha u(t) + f(t, u(t)), \quad t \in \mathbb{R},$$

has a unique almost automorphic mild solution whenever $L < l(\alpha, \rho)$.

Remark 4.4. It is interesting to note, in relation to Remark 3.8, that the behavior of the fractional powers in equation (4.1) is better for $\alpha > 1$ until some point $s_0(\rho)$ depending on the number ρ, in the sense that the Lipschitz constant L can be relaxed compared with the case $\alpha = 1$. The limit case $\alpha = 2$ is the worst, because $||\varphi_\alpha||^{-1}$ goes to zero as α goes to 2, independently of the value of ρ.

We note that if $f \in AA(X)$ and its derivative f' exists and is uniformly continuous on $\mathbb{R}$, then $f' \in AA(X)$, see [14, Theorem 2.4.1]. In the following result we use the space of all almost automorphic differentiable functions defined by

$$AA^1(X) := \{f \in AA(X) : f' \text{ exists and belongs to } AA(X)\},$$

which becomes a Banach space endowed with the norm

$$\|\eta\|_{AA(X)} := \|\eta\|_\infty + \|\eta'\|_\infty.$$

Theorem 4.5. *Let* $1 < \alpha < 2$. *Assume that* A *generates an* α*-resolvent family* $\{S_\alpha(t)\}_{t\geq 0}$ *such that*

$$\|S_\alpha(t)\| \leq \phi_\alpha(t) \text{ and } \|S'_\alpha(t)\| \leq \psi_\alpha(t), \text{ for all } t \geq 0, \text{ with } \phi_\alpha, \psi_\alpha \in L^1(\mathbb{R}_+).$$

Let $f : \mathbb{R} \times X \times X \mapsto X$ *almost automorphic in* t *for each* $x, y \in X$ *and satisfies a Lipschitz condition uniformly in* t, *that is,*

$$\|f(t, x_1, y_1) - f(t, x_2, y_2)\| \leq L_1\|x_1 - x_2\| + L_2\|y_1 - y_2\|, \text{ for all } x, y \in X,$$

with

$$L := max\{L_1, L_2\} < \frac{1}{||\psi||_1 + ||\phi||_1}.$$

Then

$$D_t^\alpha u(t) = Au(t) + f(t, u(t), u'(t)), \quad t \in \mathbb{R}, \tag{4.2}$$

has a unique almost automorphic differentiable mild solution.

Proof. We define the operator $H : AA^1(X) \mapsto AA^1(X)$ by

$$(H\varphi)(t) := \int_{-\infty}^{t} S_\alpha(t-s) f(s, \varphi(s), \varphi'(s)) ds, \quad t \in \mathbb{R}.$$

We will show H is well defined, for that first we will prove $f(\cdot, \varphi(\cdot), \varphi'(\cdot)) \in AA(X)$. Since f is almost automorphic in t then for any sequence $(s'_n) \subset \mathbb{R}$ exist a subsequence (s_n) such that

$$f(t + s_n, x, y) \to g(t, x, y) \text{ and } g(t - s_n, x, y) \to f(t, x, y) \text{ as } n \to \infty,$$

for all $t \in \mathbb{R}$. For this sequence (s_n) exist a subsequence (s_m) such that

$$\varphi(t + s_m) \to \varphi_1(t) \text{ and } \varphi_1(t - s_m) \to \varphi(t) \text{ as } m \to \infty,$$

for all $t \in \mathbb{R}$. Again for this sequence (s_m) exist a subsequence (s_l) such that

$$\varphi'(t + s_l) \to \varphi_2(t) \text{ and } \varphi_2(t - s_l) \to \varphi'(t) \text{ as } l \to \infty,$$

for all $t \in \mathbb{R}$. So we have

$$f(t+s_l,x,y) \to g(t,x,y) \quad \text{and} \quad g(t-s_l,x,y) \to f(t,x,y) \text{ as } l \to \infty, \text{ for all } t \in \mathbb{R}.$$
$$\varphi(t+s_l) \to \varphi_1(t) \quad \text{and} \quad \varphi_1(t-s_l) \to \varphi(t) \text{ as } l \to \infty, \text{ for all } t \in \mathbb{R}.$$

We define $F(t) := f(t,\varphi(t),\varphi'(t))$ and $G(t) := g(t,\varphi_1(t),\varphi_2(t))$, then

$$\begin{aligned}
\|F(t+s_l)-G(t)\| &= \|f(t+s_l,\varphi(t+s_l),\varphi'(t+s_l)) - g(t,\varphi_1(t),\varphi_2(t))\| \\
&\leq \|f(t+s_l,\varphi(t+s_l),\varphi'(t+s_l)) - f(t+s_l,\varphi_1(t),\varphi_2(t))\| \\
&+ \|f(t+s_l,\varphi_1(t),\varphi_2(t)) - g(t,\varphi_1(t),\varphi_2(t))\| \\
&\leq L_1\|\varphi(t+s_l)-\varphi_1(t)\| + L_2\|\varphi'(t+s_l)-\varphi_2(t)\| \\
&+ \|f(t+s_l,\varphi_1(t),\varphi_2(t)) - g(t,\varphi_1(t),\varphi_2(t))\| \\
&< \varepsilon,
\end{aligned}$$

when $l \to \infty$. So $F(t+s_l) \to G(t)$ when $l \to \infty$. In an analogous way we have $G(t-s_l) \to F(t)$. Then $F \in AA(X)$ and so $H\varphi \in AA(X)$ by Lemma 3.1. Furthermore

$$(H\varphi)'(t) = \int_{-\infty}^{t} S'_\alpha(t-s)f(s,\varphi(s),\varphi'(s))ds,$$

then $(H\varphi)' \in AA(X)$ by Lemma 3.1. Therefore H is well defined proving the claim.

Now we take $\varphi_1,\varphi_2 \in AA^1(X)$, then we have

$$\begin{aligned}
\|H\varphi_1 - H\varphi_2\|_\infty &= \sup_{t\in\mathbb{R}} \left\| \int_{-\infty}^{t} S(t-s)[f(s,\varphi_1(s),\varphi_1'(s)) - f(s,\varphi_2(s),\varphi_2'(s))]ds \right\| \\
&\leq \sup_{t\in\mathbb{R}} \int_{-\infty}^{t} \|S(t-s)\|[L_1||\varphi_1(s)-\varphi_2(s)|| + L_2||\varphi_1'(s)-\varphi_2'(s)||]ds \\
&\leq [L_1\|\varphi_1-\varphi_2\|_\infty + L_2\|\varphi_1'-\varphi_2'\|_\infty] \int_0^\infty |\varphi(s)|ds \\
&\leq L||\varphi_1-\varphi_2||_{AA(X)}||\varphi||_1
\end{aligned}$$

where $L = max\{L_1,L_2\}$. In similar way we have

$$\|(H\varphi_1)' - (H\varphi_2)'\|_\infty \leq L||\varphi_1-\varphi_2||_{AA(X)}||\psi||_1.$$

Then

$$\begin{aligned}
\|H\varphi_1 - H\varphi_2\|_{AA(X)} &= \|H\varphi_1 - H\varphi_2\|_\infty + \|(H\varphi_1)' - (H\varphi_2)'\|_\infty \\
&\leq L\|\varphi_1-\varphi_2\|_{AA(X)}[||\varphi||_1 + ||\psi||_1],
\end{aligned}$$

proving that H is a contraction. Hence, there exists a unique $u \in AA^1(X)$, such that $Hu = u$, i.e. $u(t) = \displaystyle\int_{-\infty}^{t} S(t-s)f(s,u(s),u'(s))ds.$ □

Corollary 4.6. *Let $1 < \alpha < 2$ and $\rho > 0$. Let $f : \mathbb{R}\times\mathbb{R}\times\mathbb{R} \mapsto \mathbb{R}$ almost automorphic in the first variable and satisfies a Lipschitz condition in the second and third variable, that is,*

$$\|f(t,x_1,y_1) - f(t,x_2,y_2)\| \leq L_1\|x_1-x_2\| + L_2\|y_1-y_2\|, \text{ for all } x,y \in \mathbb{R},$$

Then

$$D_t^\alpha u(t) = Au(t) + f(t,u(t),u'(t)), \quad t \in \mathbb{R},$$

has a unique almost automorphic differentiable mild solution whenever

$$max\{L_1, L_2\} < [\frac{2}{\alpha\rho^\alpha} - \frac{1}{\rho^\alpha} - \frac{2}{\alpha\rho^\alpha \cos(\pi/\alpha)}(1+\rho(\sin(\pi/\alpha))^2)]^{-1}.$$

Proof. The resolvent family $S_\alpha(t)$ is given by (3.5). Moreover from the proof of Corollary 3.7 there is a function $\phi_\alpha \in L^1(\mathbb{R})$ such that $\|S_\alpha(t)\| \le \phi_\alpha(t)$ and

$$||\varphi_\alpha||_1 = \frac{2}{\alpha\rho^\alpha} - \frac{1}{\rho^\alpha} - \frac{2}{\alpha\rho^\alpha \cos(\pi/\alpha)} \tag{4.3}$$

We have

$$\begin{aligned} S'_\alpha(t) &= \frac{-1}{\pi}\sin\pi\alpha \int_0^\infty e^{-rt}\frac{r^{\alpha+1}}{r^{2\alpha}+2r^\alpha\rho^\alpha\cos\pi\alpha+\rho^{2\alpha}}dr \\ &- \frac{2}{\alpha\rho^{\alpha-2}}e^{t\rho\cos\pi/\alpha}\cos[t\rho\sin\pi/\alpha]. \end{aligned}$$

Hence

$$\begin{aligned} |S'_\alpha(t)| &\le \frac{1}{\pi}|\sin\pi\alpha| \int_0^\infty e^{-rt}\frac{r^{\alpha+1}}{r^{2\alpha}+2r^\alpha\rho^\alpha\cos\pi\alpha+\rho^{2\alpha}}dr \\ &+ \frac{2}{\alpha\rho^{\alpha-2}}e^{t\rho\cos\pi/\alpha} =: \psi_\alpha(t). \end{aligned}$$

Then, applying Fubini's Theorem and taking into account that $\sin(\pi\alpha) < 0$ we obtain

$$\begin{aligned} \int_0^\infty \psi_\alpha(t)dt &= \frac{1}{\pi}|\sin\pi\alpha| \int_0^\infty\int_0^\infty e^{-rt}\frac{r^{\alpha+1}}{r^{2\alpha}+2r^\alpha\rho^\alpha\cos\pi\alpha+\rho^{2\alpha}}drdt \\ &+ \frac{2}{\alpha\rho^{\alpha-2}}\int_0^\infty e^{t\rho\cos\pi/\alpha}dt \\ &= \frac{-1}{\pi}\sin\pi\alpha \int_0^\infty e^{-rt}\frac{r^{\alpha}}{r^{2\alpha}+2r^\alpha\rho^\alpha\cos\pi\alpha+\rho^{2\alpha}}dr \\ &- \frac{2}{\alpha\rho^{\alpha-1}\cos(\pi/\alpha)} \end{aligned}$$

Since $S_\alpha(0) = 0$, from (3.5) we deduce that the integral term in the last equality is equal to $\frac{-2\cos(\pi/\alpha)}{\alpha\rho^{\alpha-1}}$. Hence

$$||\psi_\alpha||_1 = \frac{-2\cos(\pi/\alpha)}{\alpha\rho^{\alpha-1}} - \frac{2}{\alpha\rho^{\alpha-1}\cos(\pi/\alpha)}. \tag{4.4}$$

From (4.3) and (4.4) we obtain

$$\frac{1}{||\psi_\alpha||_1 + ||\phi_\alpha||_1} = [\frac{2}{\alpha\rho^\alpha} - \frac{1}{\rho^\alpha} - \frac{2}{\alpha\rho^\alpha \cos(\pi/\alpha)}(1+\rho(\sin(\pi/\alpha))^2)]^{-1}.$$

□

Remark 4.7. We observe that the function $m(\alpha,\rho) := [\frac{2}{\alpha\rho^\alpha} - \frac{1}{\rho^\alpha} - \frac{2}{\alpha\rho^\alpha\cos(\pi/\alpha)}(1+\rho(\sin(\pi/\alpha))^2)]^{-1}$ is strictly decreasing on α in the interval $(1,2)$, independently of the values of ρ. Moreover $m(1,\rho) = \rho/3$ and $m(2,\rho) = 0$. It shows that $\alpha = 1$ is the better case and $\alpha = 2$ is the worst, in contrast with the situation of Corollary 4.3.

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