ALMOST AUTOMORPHIC SOLUTIONS OF DIFFERENCE EQUATIONS

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ABSTRACT. We introduce discrete almost automorphic functions (sequences) defined on the set of integers with values in a Banach space X, extending the theory of discrete almost periodic functions. Given a bounded linear operator T defined on X and a discrete almost automorphic function $f(n)$, we give criteria for the existence of discrete almost automorphic solutions of the linear difference equation $\Delta u(n) = Tu(n) + f(n)$. We also prove the existence of a discrete almost automorphic solution of the nonlinear difference equation $\Delta u(n) = Tu(n) + g(n, u(n))$ assuming $g(n, x)$ is discrete almost automorphic in n for each $x \in X$, satisfies a global Lipschitz type condition and takes values on X.

1. INTRODUCTION

The theory of difference equations has grown at an accelerated pace in the last decades. It now occupies a central position in applicable analysis and play an important role in mathematics as a whole.

A very important aspect of the qualitative study of the solutions of difference equations is their periodicity. Periodic difference equations and systems have been treated, among others, by Agarwal and Popenda [2], Corduneanu [10], Halanay [18], Pang and Agarwal [21], Sugiyama [25], Elaydi [11] and Agarwal [1]. Almost periodicity of a discrete function was first introduced by Whalter [27, 28] and then studied by Corduneanu [10]. Recently, several papers [3, 19, 20, 22, 23, 24, 35] are devoted to study almost periodic solutions of difference equations. However, to the best of our knowledge, the concept of discrete almost automorphic functions has not been introduced in the literature until now. This motivates us to investigate their properties as well as to study discrete almost automorphic solutions of linear and nonlinear difference equations.

The theory of continuous almost automorphic functions was introduced by S. Bochner, in relation to some aspects of differential geometry [7, 5, 6, 4]. A unified and homogeneous exposition of the theory and its applications was first given by N'Guérékata in his book [16]. After that, there has been a real resurgence interest in the study of almost automorphic functions.

Important contributions to the theory of almost automorphic functions have been obtained, for example, in the papers [15, 29, 30, 31, 32, 33, 34], in the books [16, 17, 33] (concerning almost automorphic functions with values in Banach spaces), and in [26] (concerning almost automorphy on groups). Also, the theory of almost automorphic functions with values in fuzzy-number-type spaces was developed in [12] (see also Chapter 4 in [17]). Recently, in [13]

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and [14], the theory of almost automorphic functions with values in a locally convex space (Fréchet space) and a p-Fréchet space has been developed.

The range of applications of almost automorphic functions include at present linear and non-linear evolution equations, integro-differential and functional-differential equations, dynamical systems, etc. A recent reference is the book [17].

This paper is organized as follows. In Section 2, we present the definition of discrete almost automorphic functions (sequences) and give some basic and related properties for our purposes. In Section 3, we discuss the existence of almost automorphic solutions of first order linear difference equations. In Section 4, we discuss the existence of almost automorphic solutions of nonlinear difference equations of the form $\Delta u(n) = Tu(n) + g(n, u(n))$, where T is a bounded operator defined on a Banach space X.

2. The basic theory

Let X be a real or complex Banach space. We recall that a function $f : \mathbb{Z} \to X$ is said to be discrete almost periodic if for any positive ϵ there exists a positive integer $N(\epsilon)$ such that any set consisting of N consecutive integers contains at least one integer p with the property that

$$||f(k+p) - f(p)|| < \epsilon, \quad k \in \mathbb{Z}.$$

In the above definition p is called an $\epsilon-$ almost period of $f(k)$ or an $\epsilon-$ translation number. We denote by $AP_d(X)$ the set of discrete almost periodic functions.

Bochner's criterion: f is a discrete almost periodic function if and only if

(N) for any integer sequence (k'_n), there exist a subsequence (k_n) such that $f(k+k_n)$ converges uniformly on $\mathbb{Z}$ as $n \to \infty$. Furthermore, the limit sequence is also a discrete almost periodic function.

The proof can be found in [9, Theorem 1.26, pp. 45-46]. Observe that functions with the property **(N)** are also called *normal* in the literature (cf. [1, p.72] or [9]).

The above characterization, as well as the definition of continuous almost automorphic functions (cf. [16]) motivates the following definition.

Definition 2.1. Let X be a (real or complex) Banach space. A function $f : \mathbb{Z} \to X$ is said to be discrete almost automorphic if for every integer sequence (k'_n), there exist a subsequence (k_n) such that

$$\lim_{n\to\infty} f(k+k_n) =: \overline{f}(k)$$

is well defined for each $k \in \mathbb{Z}$ and

$$\lim_{n\to\infty} \overline{f}(k-k_n) = f(k)$$

for each $k \in \mathbb{Z}$.

Remark 2.2.

(i) If f is a continuous almost automorphic function in $\mathbb{R}$ then $f\mid_{\mathbb{Z}}$ is discrete almost automorphic.

(ii) If the convergence in Definition 2.1 is uniform on $\mathbb{Z}$, then we get discrete almost periodicity.

We denote by $AA_d(X)$ the set of discrete almost automorphic functions. Such as the continuous case we have that discrete almost automorphicity is a more general concept than discrete almost periodicity, that is

$$AP_d(X) \subset AA_d(X).$$

Example 2.3. *Let us consider the function $f : \mathbb{Z} \to \mathbb{R}$ defined by*

$$f(k) := \frac{1}{2 + \cos(k) + \cos(\sqrt{2}k)}$$

this function is discrete almost automorphic since it is the restriction to $\mathbb{Z}$ of the almost automorphic function $f(t) := \frac{1}{2+\cos(t)+\cos(\sqrt{2}t)}$ (see e.g. [8]). We will see that it is not discrete almost periodic. In fact, let N be a nonnegative integer and consider the set $I := \{1, 2, 3, 4, ..., N\}$. We first note that $f(0+p)-f(0) > 0$ for all $p \in I$, since $\cos(s)+\cos(\sqrt{2}s) < 2$. Now let $g : I \to \mathbb{R}$ be defined by $g(p) := f(p) - f(0)$. It is clear that there is a $k_0 \in I$ such that $g(k_0) \leq g(p)$ for all p en I. Let $\varepsilon := g(k_0) > 0$, then for $k = 0$ we have

$$|f(0 + p) - f(0)| = |g(p)| \geq \varepsilon, \forall\, p \in I.$$

Therefore f is not discrete almost periodic.

Discrete almost automorphic functions have the following fundamental properties.

Theorem 2.4. *Let u, v be discrete almost automorphic functions; then the following assertions are valid*

(i) *$u + v$ is discrete almost automorphic;*
(ii) *cu is discrete almost automorphic for every scalar c;*
(iii) *For each fixed l in $\mathbb{Z}$, the function $u_l : \mathbb{Z} \to X$ defined by $u_l(k) := u(k + l)$ is discrete almost automorphic;*
(iv) *The function $\hat{u} : \mathbb{Z} \to X$ defined by $\hat{u}(k) := u(-k)$ is discrete almost automorphic;*
(v) *$\sup_{k\in\mathbb{Z}} \|u(k)\| < \infty$, that is, u is a bounded function;*
(vi) *$\sup_{k\in\mathbb{Z}} \|\overline{u}(k)\| \leq \sup_{k\in\mathbb{Z}} \|u(k)\|$, where*

$$\lim_{n\to\infty} u(k + k_n) = \overline{u}(k) \text{ and } \lim_{n\to\infty} \overline{u}(k - k_n) = u(k).$$

Proof. The proof of all statements follows the same lines as in the continuous case (see [16, Theorem 2.1.3]) and therefore is omitted. □

As a consequence of the above theorem, the space of discrete almost automorphic functions provided with the norm

$$\|u\|_d := \sup_{k\in\mathbb{Z}} \|u(k)\|,$$

becomes a Banach space. The proof is straightforward and therefore omitted.

Theorem 2.5. *Let X, Y be Banach spaces and $u : \mathbb{Z} \to X$ an discrete almost automorphic function. If $\phi : X \to Y$ is a continuous function, then the composite function $\phi \circ u : \mathbb{Z} \to Y$ is discrete almost automorphic.*

Proof. Let (k'_n) a sequence in $\mathbb{Z}$, since $u \in AA_d(X)$ there exist a subsequence (k_n) of (k'_n) such that $\lim_{n\to\infty} u(k + k_n) = v(k)$ is well defined for each $k \in \mathbb{Z}$ and $\lim_{n\to\infty} v(k - k_n) = u(k)$ for each $k \in \mathbb{Z}$. Since ϕ is continuous, we have $\lim_{n\to\infty} \phi(u(k + k_n)) = \phi(\lim_{n\to\infty} u(k + k_n)) =$

$\phi(v(k))$. In similar way, we have $\lim_{n\to\infty} \phi(v(k-k_n)) = \phi(\lim_{n\to\infty} v(k-k_n)) = \phi(u(k))$, therefore $\phi \circ u$ is in $AA_d(Y)$. □

Corollary 2.6. *If A is a bounded linear operator in X and $u : \mathbb{Z} \to X$ a discrete almost automorphic function, then $Au(k), k \in \mathbb{Z}$ is also discrete almost automorphic.*

Theorem 2.7. *Let $u : \mathbb{Z} \to \mathbb{C}$ and $f : \mathbb{Z} \to X$ be discrete almost automorphic. Then $uf : \mathbb{Z} \to X$ defined by $(uf)(k) = u(k)f(k), k \in \mathbb{Z}$ is also discrete almost automorphic.*

Proof. Let (k'_n) a sequence in $\mathbb{Z}$. There exist a subsequence (k_n) of (k'_n) such that $\lim_{n\to\infty} u(k+k_n) = \overline{u}(k)$ is well defined for each $k \in \mathbb{Z}$ and $\lim_{n\to\infty} \overline{u}(k-k_n) = u(k)$ for each $k \in \mathbb{Z}$. Also we have $\lim_{n\to\infty} f(k+k_n) = \overline{f}(k)$ is well defined for each $k \in \mathbb{Z}$ and $\lim_{n\to\infty} \overline{f}(k-k_n) = f(k)$ for each $k \in \mathbb{Z}$. The proof now follows from Theorem 2.4 and the identities

$$u(k+k_n)f(k+k_n) - \overline{u}(k)\overline{f}(k) = u(k+k_n)(f(k+k_n) - \overline{f}(k)) + (u(k+k_n) - \overline{u}(k))\overline{f}(k),$$

and

$$\overline{u}(k-k_n)\overline{f}(k-k_n) - u(k)f(k) = \overline{u}(k-k_n)(\overline{f}(k-k_n) - f(k)) + (\overline{u}(k-k_n) - u(k))f(k),$$

valid for all $k \in \mathbb{Z}$. □

For applications to nonlinear difference equations the following definition, of discrete almost automorphic function depending on one parameter, will be useful.

Definition 2.8. A function $u : \mathbb{Z} \times X \to X$ is said to be discrete almost automorphic in k for each $x \in X$, if for every sequence of integers numbers (k'_n), there exist a subsequence (k_n) such that

$$\lim_{n\to\infty} u(k+k_n, x) =: \overline{u}(k, x)$$

is well defined for each $k \in \mathbb{Z}$, $x \in X$, and

$$\lim_{n\to\infty} \overline{u}(k-k_n, x) = u(k, x)$$

for each $k \in \mathbb{Z}$ and $x \in X$.

The proof of the following result is omitted (see [16, Section 2.2]).

Theorem 2.9. *If $u, v : \mathbb{Z} \times X \to X$ are discrete almost automorphic functions in k for each x in X, the following are true*

(i) *$u + v$ is discrete almost automorphic in k for each x in X.*
(ii) *cu is discrete almost automorphic in k for each x in X, where c is an arbitrary scalar.*
(iii) $\sup_{k\in\mathbb{Z}} \|u(k, x)\| = M_x < \infty$, *for each x in X.*
(iv) $\sup_{k\in\mathbb{Z}} \|\overline{u}(k, x)\| = N_x < \infty$, *for each x in X, where $\overline{u}$ is the function in Definition 2.8.*

The following result will be used to study almost automorphy of solution of nonlinear difference equations.

Theorem 2.10. *Let $f : \mathbb{Z} \times X \to X$ be discrete almost automorphic in k for each x in X, and satisfies a Lipschitz condition in x uniformly in k, that is*

$$\|f(k, x) - f(k, y)\| \le L\|x - y\|, \text{ for all } x, y \in X.$$

Suppose $\varphi : \mathbb{Z} \to X$ is discrete almost automorphic, then the function $U : \mathbb{Z} \to X$ defined by $U(k) = u(k, \varphi(k))$ is discrete almost automorphic.

Proof. Let (k'_n) be a sequence in $\mathbb{Z}$. There exist a subsequence (k_n) of (k'_n) such that $\lim_{n\to\infty} f(k+k_n, x) = \overline{f}(k,x)$ for all $k \in \mathbb{Z}, x \in X$ and $\lim_{n\to\infty} \overline{f}(k-k_n, x) = f(k,x)$ for each $k \in \mathbb{Z}, x \in X$. Also we have $\lim_{n\to\infty} \varphi(k+k_n) = \overline{\varphi}(k)$ is well defined for each $k \in \mathbb{Z}$ and $\lim_{n\to\infty} \overline{\varphi}(k-k_n) = \varphi(k)$ for each $k \in \mathbb{Z}$. Since the function u is Lipschitz, using the identities

$$f(k+k_n, \varphi(k+k_n)) - \overline{f}(k, \overline{\varphi}(k)) = f(k+k_n, \varphi(k+k_n)) - f(k+k_n, \overline{\varphi}(k)) + f(k+k_n, \overline{\varphi}(k)) - \overline{f}(k, \overline{\varphi}(k))$$

and

$$\overline{f}(k-k_n, \overline{\varphi}(k-k_n)) - f(k, \varphi(k)) = \overline{f}(k-k_n, \overline{\varphi}(k-k_n)) - \overline{f}(k-k_n, \varphi(k)) + \overline{f}(k-k_n, \varphi(k)) - f(k, \varphi(k)),$$

valid for all $k \in \mathbb{Z}$, we get the desired proof. □

We will denote $AA_d(\mathbb{Z} \times X)$ the space of the discrete almost automorphics functions in $k \in \mathbb{Z}$, for each x in X.

Let Δ denote the forward difference operator of the first order, i.e. for each $u : \mathbb{Z} \to X$, and $n \in \mathbb{Z}$, $\Delta u(n) = u(n+1) - u(n)$.

Theorem 2.11. *Let $\{u(k)\}_{k\in\mathbb{Z}}$ be a discrete almost automorphic function, then $\Delta u(k)$ is also discrete almost automorphic.*

Proof. Since $\Delta u(k) = u(k+1) - u(k)$, then by (i) and (iii) in Theorem 2.4, we have that $\Delta u(k)$ is discrete almost automorphic. □

The following result will be the key in the study of discrete almost automorphic solutions of linear and nonlinear difference equations.

Theorem 2.12. *Let $v : \mathbb{Z} \to \mathbb{C}$ be a summable function, i.e.*

$$\sum_{k\in\mathbb{Z}} |v(k)| < \infty.$$

Then for any discrete almost automorphic function $u : \mathbb{Z} \to X$ the function $w(k)$ defined by

$$w(k) = \sum_{l\in\mathbb{Z}} v(l)u(k-l), \quad k \in \mathbb{Z}$$

is also discrete almost automorphic.

Proof. Let (k'_n) be a arbitrary sequence of integers numbers. Since u is discrete almost automorphic there exists a subsequence (k_n) of (k'_n) such that

$$\lim_{n\to\infty} u(k+k_n) = \overline{u}(k)$$

is well defined for each $k \in \mathbb{Z}$ and

$$\lim_{n\to\infty} \overline{u}(k-k_n) = u(k)$$

for each $k \in \mathbb{Z}$. Note that

$$\|w(k)\| \le \sum_{l\in\mathbb{Z}} \|v(l)\|\|u(k-l)\| \le \sum_{l\in\mathbb{Z}} \|v(l)\|\|u\|_d < \infty,$$

then, by Lebesgue's dominated convergence theorem, we obtain

$$\lim_{n\to\infty} w(k+k_n) = \sum_{l\in\mathbb{Z}} v(l) \lim_{n\to\infty} u(k+k_n-l) = \sum_{l\in\mathbb{Z}} v(l)\overline{u}(k-l) =: \overline{w}(k).$$

In similar way, we prove

$$\lim_{n\to\infty} \overline{w}(k-k_n) = w(k),$$

and then w is discrete almost automorphic. □

Remark 2.13.

(i) The same conclusions of the previous results holds in case of the finite convolution

$$w(k) = \sum_{l=0}^{k} v(k-l)u(l), \quad k \in \mathbb{Z}$$

and the convolution

$$w(k) = \sum_{l=-\infty}^{k} v(k-l)u(l), \quad k \in \mathbb{Z}.$$

(ii) The results are true in case of consider an operator valued function $v : \mathbb{Z} \to \mathcal{B}(X)$ such that

$$\sum_{k\in\mathbb{Z}} ||v(k)|| < \infty.$$

A typical example is $v(k) = T^k$, where $T \in \mathcal{B}(X)$ satisfies $||T|| < 1$.

3. Almost automorphic solutions of first order linear difference equations

Difference equations usually describes the evolution of certain phenomena over the course of time. In this section we deal with those equations known as the first-order linear difference equations. These equations naturally apply to various fields, like biology (the study of competitive species in population dynamics), physics (the study of motions of interacting bodies), the study of control systems, neurology, and electricity, see [11, Chapter 3].

We are interested in finding discrete almost automorphic solutions of the following system of first order linear difference equations, written in vector form

$$\Delta u(n) = Tu(n) + f(n) \tag{3.1}$$

where T is a matrix or, more generally, a bounded linear operator defined on a Banach space X and f is in $AA_d(X)$. Note that equation (3.1) is equivalent to

$$u(n+1) = Au(n) + f(n), \tag{3.2}$$

where $A = I + T$. We begin studying the scalar case. We denote $\mathbb{D} := \{z \in \mathbb{C} : |z| = 1\}$.

Theorem 3.1. *Let X be a Banach space. If $A := \lambda \in \mathbb{C} \setminus \mathbb{D}$ and $f : \mathbb{Z} \to X$ is discrete almost automorphic, then there is a discrete almost automorphic solution of (3.2) given by*

(i) $u(n) = \sum_{k=-\infty}^{n} \lambda^{n-k} f(k-1)$ *in case* $|\lambda| < 1$; *and*

(ii) $u(n) = -\sum_{k=n}^{\infty} \lambda^{n-k-1} f(k)$ *in case* $|\lambda| > 1$.

Proof.

(i) Define $v(k) = \lambda^k$. Then $v \in \ell^1(\mathbb{Z})$ and hence, by Theorem 2.12, we obtain $u \in AA_d(X)$. Next, we note that u is solution of (3.2) because

$$u(n+1) = \sum_{k=-\infty}^{n+1} \lambda^{n+1-k} f(k-1) = \sum_{k=-\infty}^{n} \lambda^{n+1-k} f(k-1) + f(n) = \lambda u(n) + f(n).$$

(ii) Define $v(k) = \lambda^{-k}$ and since $|\lambda| > 1$ we have $v \in \ell^1(\mathbb{Z})$. It follows, by Theorem 2.12, that $u \in AA_d(X)$. Finally, we check that u is solution of (3.2) as follows

$$\begin{aligned} u(n+1) &= -\sum_{k=n+1}^{\infty} \lambda^{n-k} f(k) = -\left(\sum_{k=n}^{\infty} \lambda^{n-k} f(k) - f(n)\right) = -\lambda \sum_{k=n}^{\infty} \lambda^{n-k-1} f(k) + f(n) \\ &= \lambda u(n) + f(n). \end{aligned}$$

□

As a consequence of the previous theorem, we obtain the following result in case of a matrix A.

Theorem 3.2. *Suppose A is a constant $n \times n$ matrix with eigenvalues $\lambda \notin \mathbb{D}$. Then for any function $f \in AA_d(\mathbb{C}^n)$ there is a discrete almost automorphic solution of (3.2).*

Proof. It is well known that there exists a nonsingular matrix S such that $S^{-1}AS = B$ is an upper triangular matrix. In (3.2) we use now the substitution $u(k) = Sv(k)$ to obtain

$$v(k+1) = Bv(k) + S^{-1}f(k), \quad k \in \mathbb{Z}. \tag{3.3}$$

Obviously, the system (3.3) is of the form as (3.2) with $S^{-1}f(k)$ a discrete almost automorphic function. The general case of an arbitrary matrix A can now be reduced to the scalar case. Indeed, the last equation of the system (3.3) is of the form

$$z(k+1) = \lambda z(k) + c(k), \quad k \in \mathbb{Z} \tag{3.4}$$

where λ is a complex number and $c(k)$ is a discrete almost automorphic function. Hence, all we need to show is that any solution $z(k)$ of (3.4) is discrete almost automorphic. But this is the content of Theorem 3.1. It then imply that the nth component $v_n(k)$ of the solution $v(k)$ of (3.3) is discrete almost automorphic. Then substituting $v_n(k)$ in the $(n-1)$th equation of (3.3) we obtain again an equation of the form (3.4) for $v_{n-1}(k)$, and so on. The proof is complete. □

As an application of the above Theorem and [1, Theorem 5.2.4] we obtain the following Corollary.

Corollary 3.3. *Assume that A is a constant $n \times n$ matrix with eigenvalues $\lambda \notin \mathbb{D}$, and suppose that $f \in AA_d(\mathbb{C}^n)$ is such that*

$$||f(k)|| \leq c\eta^{|k|}$$

for all large k, where $c > 0$ and $\eta < 1$. Then there is a discrete almost automorphic solution $u(k)$ of (3.2), which satisfies

$$||u(k)|| \leq c\nu^{|k|},$$

for some $\nu > 0$.

We can replace $\lambda \in \mathbb{C}$ in Theorem 3.1 by a general bounded operator $A \in \mathcal{B}(X)$, and use (ii) of Remark 2.13 in the proof of the first part of Theorem 3.1, to obtain the following result.

Theorem 3.4. *Let X be a Banach space and $A \in \mathcal{B}(X)$ such that $||A|| < 1$. Let $f \in AA_d(X)$. Then there is a discrete almost automorphic solution of (3.2).*

We can also prove the following result.

Theorem 3.5. *Let X be a Banach space. Suppose $f \in AA_d(X)$ and $A = \sum_{k=1}^{N} \lambda_k P_k$ where the complex numbers λ_k are mutually distinct with $|\lambda_k| \neq 1$, and $(P_k)_{1 \leq k \leq N}$ forms a complex system $\sum_{k=1}^{N} P_k = I$ of mutually disjoint projections on X. Then equation (3.2) admits a discrete almost automorphic solution.*

Proof. Let $k \in \{1, ..., N\}$ be fixed. Applying the projection P_k to equation (3.2) we obtain

$$P_k u(n+1) = P_k A u(n) + P_k f(n) = \lambda_k P_k u(n) + P_k f(n).$$

By Corollary 2.6 we have $P_k f \in AA_d(X)$, since P_k is bounded. Therefore, by Theorem 3.1, we get $P_k u \in AA_d(X)$. We conclude that $u(n) = \sum_{k=1}^{N} P_k u(n) \in AA_d(X)$ as a finite sum of discrete almost periodic functions. □

We finish this section with the following simple example concerning the heat equation (cf. [11, p.157]).

Example 3.6. *Consider the distribution of heat through a thin bar composed by a homogeneous material. Let $x_1, x_2, ..., x_k$ be k equidistant points on the bar. Let $T_i(n)$ be the temperature at time $t_n = (\Delta t)n$ at the point x_i, $1 \leq i \leq k$. Under certain conditions one may derive the equation*

$$T(n+1) = AT(n) + f(n), \quad n \in \mathbb{Z} \tag{3.5}$$

where the vector $T(n)$ consist of the components $T_i(n)$, $1 \leq i \leq k$, and A is a tridiagonal Toeplitz matrix. Its eigenvalues may be found by the formula

$$\lambda_n = (1 - 2\alpha) + \alpha \cos(\frac{n\pi}{k+1}), \quad n = 1, 2..., k$$

where α is a constant of proportionality concerning the difference of temperature between the a point x_i and the nearby points x_{i-1} and x_{i+1} (see [11]). Assuming

$$0 < \alpha < 1/2$$

we obtain $|\lambda| < 1$ for all eigenvalues λ of A. Theorem 3.4 then implies that for each $f \in AA_d(\mathbb{C}^k)$ there is a discrete almost automorphic solution of (3.5).

4. Almost automorphic solutions of semilinear difference equations

We want to find conditions under which is possible to find discrete almost automorphic solutions to the equation

$$u(n+1) = Au(n) + f(n, u(n)), \quad n \in \mathbb{Z}, \tag{4.1}$$

where A is a bounded linear operator defined on a Banach space X and $f \in AA_d(\mathbb{Z} \times X)$.

Our main result in this section is the following theorem for the scalar case.

Theorem 4.1. *Let $A := \lambda \in \mathbb{C} \setminus \mathbb{D}$ and $f : \mathbb{Z} \times X \to X$ discrete almost automorphic in k for each $x \in X$. Suppose that f satisfies the following Lipschitz type condition*

$$\|f(k, x) - f(k, y)\| \leq L\|x - y\|, \text{ for all } x, y \in X \text{ and } k \in \mathbb{Z}. \tag{4.2}$$

Then equation (4.1) have a unique discrete almost automorphic solution satisfying

(i) $u(n)=\sum_{k=-\infty}^{n}\lambda^{n-k}f(k-1,u(k-1))$ *in case* $|\lambda|<1-L$ *and*

(ii) $u(n)=-\sum_{k=n}^{\infty}\lambda^{n-k-1}f(k,u(k))$ *in case* $|\lambda|>1+L$.

Proof. Case $|\lambda|<1-L$: We define the operator $F: AA_d(X)\to AA_d(X)$, by

$$F(\varphi)(n)=\sum_{k=-\infty}^{n}\lambda^{n-k}f(k-1,\varphi(k-1)),\quad n\in\mathbb{Z}.$$

Since $\varphi\in AA_d(X)$ and $f(k,x)$ satisfies (4.2), we obtain by Theorem 2.10 that $f(\cdot,\varphi(\cdot))$ is in $AA_d(X)$. So F is well-defined thanks to Theorem 2.12. Now, given $u_1,u_2\in AA_d(X)$, we have

$$\begin{aligned}\|F(u_1)-F(u_2)\|_d &\le \sup_{n\in\mathbb{Z}}\sum_{k=-\infty}^{n}|\lambda|^{n-k}\|f(k-1,u_1(k-1))-f(k-1,u_2(k-1))\|\\ &\le \sup_{n\in\mathbb{Z}}\sum_{k=-\infty}^{n}|\lambda|^{n-k}L\|u_1(k-1)-u_2(k-1)\|\\ &\le L\|u_1-u_2\|_d\sup_{n\in\mathbb{Z}}\sum_{k=-\infty}^{n}|\lambda|^{n-k}=L\|u_1-u_2\|_d\sup_{n\in\mathbb{Z}}\sum_{j=0}^{\infty}|\lambda|^{j}\\ &\le L\|u_1-u_2\|_d\frac{1}{1-|\lambda|}.\end{aligned}$$

Since $|\lambda|<1-L$ we obtain that the function F is a contraction. Then there exist an unique function u in $AA_d(X)$ such that $Fu=u$. That is, u satisfies $u(n)=\sum_{k=-\infty}^{n}\lambda^{n-k}f(k-1,u(k-1))$ and hence u is solution of equation (4.1) (cf. the proof of (i) in Theorem 3.1).

Case $|\lambda|>1+L$: We define $F: AA_d(X)\to AA_d(X)$, by

$$F(\varphi)(n)=-\sum_{k=n}^{\infty}\lambda^{n-k-1}f(k,\varphi(k)),\quad n\in\mathbb{Z},$$

and with similar arguments as in the previous case we obtain that F is well-defined. Now, given $u_1, u_2 \in AA_d(X)$, we have

$$\begin{aligned}\|F(u_1) - F(u_2)\|_d &\leq \sup_{n\in\mathbb{Z}} \sum_{k=n}^{\infty} |\lambda|^{n-k-1} \|f(k, u_1(k)) - f(k, u_2(k))\| \\ &\leq \sup_{n\in\mathbb{Z}} \sum_{k=n}^{\infty} |\lambda|^{n-k-1} L\|u_1(k-1) - u_2(k-1)\| \\ &\leq L\|u_1 - u_2\|_d \sup_{n\in\mathbb{Z}} \sum_{k=n}^{\infty} |\lambda|^{n-k-1} \\ &= L\|u_1 - u_2\|_d \sup_{n\in\mathbb{Z}} \sum_{j=0}^{\infty} |\lambda^{-1}|^{j+1} \quad (\text{ taking } j = k-n) \\ &\leq L\|u_1 - u_2\|_d \frac{1}{|\lambda| - 1}.\end{aligned}$$

Therefore F is a contraction and then there exist an unique function $u \in AA_d(X)$ such that $Fu = u$. The function u satisfies

$$u(n) = -\sum_{k=n}^{\infty} \lambda^{n-k-1} f(k, u(k)), \quad n \in \mathbb{Z},$$

and hence is a solution of equation (4.1) (cf. the proof of (ii) in Theorem 3.1).

□

In the particular case $f(k, x) := h(k)g(x)$ we obtain the following corollary.

Corollary 4.2. *Let $A := \lambda \in \mathbb{C} \setminus \mathbb{D}$. Suppose that g satisfies a Lipschitz condition*

$$\|g(x) - g(y)\| \leq L\|x - y\|, \text{ for all } x, y \in X. \tag{4.3}$$

Then for each $h \in AA_d(X)$, equation (4.1) have a unique discrete almost automorphic solution whenever $|\lambda| < 1 - L||h||_d$ or $|\lambda| > 1 + L||h||_d$.

The case of a bounded operator A can be treated assuming extra conditions on the operator. The proof of the next result follows the same lines of the first part in the proof of Theorem 4.1, using (ii) of Remark 2.13.

Theorem 4.3. *Let $A \in \mathcal{B}(X)$ and suppose that $f \in AA_d(\mathbb{Z} \times X)$ is such that*

$$\|f(k, x) - f(k, y)\| \leq L\|x - y\|, \text{ for all } x, y \in X \text{ and } k \in \mathbb{Z}. \tag{4.4}$$

Then equation (4.1) have a unique discrete almost automorphic solution whenever $||A|| < 1-L$.

5. Conclusion and Future Directions

This paper is the started point to research discrete almost automorphic functions. The aim is to present for the first time a brief exposition of the theory and its application to the field of difference equations in abstract spaces. We first state, for future reference, several results which can be directly deduced from the continuous case and then, we analyze the existence of discrete almost automorphic solutions of linear and nonlinear difference equations in the scalar and in the abstract setting. Many questions remain open, as for example to state explicitly the case $|\lambda| = 1$ in Theorem 3.1, which should require an additional condition on the Banach space X (cf. [16, Theorem 2.4.6]) and thus to extend Corduneanu's Theorem to the abstract Banach space setting (see [1, Theorem 2.10.1, p.73]). Another question of interest is to prove the converse of (i) in Remark 2.2, that is, assuming that $u(n)$ is a discrete almost automorphic function, to find an almost automorphic function $f(t), (t \in \mathbb{R})$ such that $u(n) = f(n)$ for all $n \in \mathbb{Z}$ (see [9, Theorem 1.27] in the almost periodic case). Concerning almost automorphic solutions of difference equations, it remain to study discrete almost automorphic solutions of Volterra difference equations as well as discrete almost automorphic solutions of functional difference equations with infinite delay. This topics should be handled by looking at the recent papers of Song [22, 23].

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