ATTRACTIVITY FOR FUNCTIONAL VOLTERRA INTEGRAL EQUATIONS OF CONVOLUTION TYPE

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Abstract. In this paper we investigate the existence of attractive and uniformly locally attractive solutions for a functional nonlinear integral equation with a general kernel. We use methods and techniques of fixed point theorems and properties of measure of noncompactness. We extend and generalize results obtained by other authors in the context of fractional functional differential equations.

1. Introduction

During the past years the theory of functional Volterra integral equations have undergone rapid development. The growth has been strongly promoted by the large number of applications that this theory has found in physics, engineering, and biology [22].

In this paper we study attractive and uniformly locally attractive solutions for the following functional Volterra integral equation of convolution type

$$u(t) = \begin{cases} \phi(t_0) + \displaystyle\int_{t_0}^{t} a(t-s) f(s, u_s)\, ds & t > t_0, \\ \phi(t) & t_0 - \sigma \leq t \leq t_0. \end{cases} \tag{1.1}$$

where $a \in L^1_{loc}(\mathbb{R}_+)$ is an scalar kernel, $\phi \in C([t_0 - \sigma, t_0], \mathbb{R})$, $f : [t_0, \infty) \times C([-\sigma, 0], \mathbb{R}) \to \mathbb{R}$ and $u_s : [-\sigma, 0] \to \mathbb{R}$ is defined by $u_s(\tau) := u(s + \tau)$ for all $\tau \in [-\sigma, 0]$.

Nonlinear integral equations have been studied extensively in the literature as regard to several qualitative aspects of the solutions. This include existence and uniqueness, perturbation [22, Chapter 11], differentiability [22, Chapter 11], maximal and minimal solutions [22, Chapter 13] and asymptotic behavior [22, Chapter 15] among others.

However, the study of existence of attractive solutions for integral equations has been little exploited. Some research in this topic are, for instance, the study of local attractivity of solutions of a nonlinear quadratic Volterra integral equation of fractional order [1], locally attractive solutions for quasi-periodically forced systems [16], global attractivity of solutions of a nonlinear functional integral equation [5], functional integral equations [7], [8] and fractional integro-differential equations involving Riemann–Liouville and Caputo fractional calculus [17], [24], [10].

The equation (1.1) was studied by F. Chen and Y. Zhou in [18] and (joint with J.J. Nieto) in [17] for the case $a(t) = \frac{t^{\alpha-1}}{\Gamma(\alpha)}$ for $0 < \alpha < 1$. Recently, J. Losada et al. in [24] proved the existence of attractive solutions on the Fréchet space $C([t_0 - \sigma, \infty), \mathbb{R})$ and the existence of uniformly locally attractive solutions in the Banach space $BC([t_0 - \sigma, \infty), \mathbb{R})$. In this case, the

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authors took $a(t) = \sum_{i=0}^{m} \frac{t^{\alpha-\alpha_{i-1}-1}}{\Gamma(\alpha-\alpha_{i-1})}$. Other works on attractivity of solutions in the context of fractional functional differential equations appear in [2, 12, 13, 14, 17, 23, 27] among others.

The starting point of our research in this paper, is the above observation that some of these fractional problems are particular cases of functional integral equations of the type (1.1).

Then it is natural to ask for the following problem: For which class of kernels $a(t)$ there exist attractive and uniformly locally attractive solutions for (1.1)?.

We have success to solve this problem for the class of kernels $a(t)$ that satisfy the following condition: There exists $0 < \beta < 1$ such that $a \in L^{\frac{1}{\beta}}([0,\infty),\mathbb{R})$ and

$$\lim_{h\to 0}\int_0^t |a(\tau+h)-a(\tau)|^{\frac{1}{1-\beta}}\,d\tau = 0, \quad t>0, \tag{1.2}$$

holds.

It is interesting to note that the class of kernels defined by $a(t) = \frac{t^{\alpha-1}}{\Gamma(\alpha)}e^{-\delta t}$ satisfy our assumptions under some mild restrictions on the parameters $\alpha > 0$ and $\delta > 0$. This class of kernels naturally appears in the theory of integral equations of convolution type. They corresponds to power type materials in the case $1 < \alpha < 2$ and $\delta = 0$ [28, Chapter 5, p.131] (creep compliance), and to Maxwell type materials in the case of $\alpha = 1$ and $\delta > 0$ (relaxation modulus), see [25, Chapter 2, section 2.4]. Some concrete examples are provided in the last section of this paper.

In order to obtain our results, we first show the existence of attractive solutions using the Schauder fixed point theorem and Carathéodory type conditions on f, among other conditions. Second, we use the properties of measure of noncompactness and a consequence of Darbo-Sadovskii fixed point theorem (see [4, 6]) to prove the existence of uniformly locally attractive solutions under a locally Lipschitz type condition on the nonlinear term f.

The outline of this paper is as follows: Section 2 is devoted to preliminaries, recalling the definitions of attractive and uniformly locally attractive solutions. Let us notice that the concept of uniform local attractivity of solutions is equivalent to the concept of asymptotic stability introduced in [15]. We also recall the Schauder fixed point theorem in the context of Hausdorff topological vector spaces. After remind the notion of measure of noncompactness, we mention a generalization of Darbo fixed point theorem due to Aghajani, Banás and Sabzali [4, Theorem 2.2]. See also the book [11] for and updated reference on the theory of measures of noncompactness and their applications.

Section 3 shows the existence of at least one attractive solution of (1.1) on the Fréchet space $C([t_0-\sigma,\infty),\mathbb{R})$ by using an application of Schauder fixed point theorem together with three hypothesis on the kernel $a(t)$ and the function f (A Carathéodory type condition of f and condition (1.2) on $a(t)$). The attractivity is ensured by means of one additional hypothesis about the existence of a positive and strictly decreasing function $\mathcal{H}$ that controls the size of the solution.

In Section 4, we prove the existence of at least one uniformly locally attractive solution for the problem (1.1), this time on the Banach space $BC([t_0-\sigma,\infty),\mathbb{R})$, by using arguments of measure of noncompactness, a consequence of Darbo-Sadovskii fixed point theorem and, again, certain hypothesis on the kernel and the function f.

Finally, in Section 5, we give some concrete examples of our main results considering as external nonlinear force the functions $f(s,\psi) = e^{-\gamma s}e^{-\sin\psi(-1)}$ and $g(s,\psi) = \frac{s^2}{1+s^4}\ln(1+|\psi(-\pi/2)|)$ as well as the coupling of the Maxwell and power type materials in the kernel $a(t) = \frac{t^{\alpha-1}}{\Gamma(\alpha)}e^{-\delta t}$. It is interesting to observe that for this function f the existence of at least one attractive solution on $C([t_0-\sigma,\infty),\mathbb{R})$ is ensured by means of the hypothesis $\delta > \gamma > 0$, whereas that for g this existence is granted in the space $BC([t_0-\sigma,\infty),\mathbb{R})$ by means of the condition $\delta > 1$.

2. Preliminaries

Let $t_0 \in \mathbb{R}$, $\sigma > 0$ be given and

$$C([t_0-\sigma,\infty),\mathbb{R}) := \{u : [t_0-\sigma,\infty) \to \mathbb{R} : u \text{ is continuous}\}.$$

We know that $C([t_0-\sigma,\infty),\mathbb{R})$ equipped with the seminorms

$$\|u\|_n = \sup_{t\in[t_0-\sigma,n]} |u(t)| \quad (n \geq 1)$$

is a Fréchet space. We recall the following facts:

(A1) A sequence $(x_n)_{n\in\mathbb{N}}$ is convergent to x in $C([t_0-\sigma,\infty),\mathbb{R})$ if and only if is uniformly convergent to x on compact subsets of $[t_0-\sigma,\infty)$. (Consequence of the definition of Fréchet's spaces).

(A2) A family $\mathfrak{F} \subset C([t_0-\sigma,\infty),\mathbb{R})$ is relatively compact if and only if for each $R > t_0-\sigma$, the restrictions to $[t_0-\sigma,R]$ of all functions from $\mathfrak{F}$ form an equicontinuous and uniformly bounded set. (Consequence of [26, Lemma 3.3]).

Definition 2.1 ([18])**.** *A solution $u(t)$ of the Equation* (1.1) *is attractive if there exist a constant b_0 that depends of t_0 such that*

$$|\phi(s)| \leq b_0 \quad \textit{for all} \;\; t_0-\sigma \leq s \leq t_0 \qquad \textit{implies} \qquad \lim_{t\to\infty} u(t) = 0.$$

Let $BC(I,\mathbb{R})$ be the space of all continuous and bounded real valued functions defined on $I := [t_0-\sigma,\infty)$ with the usual norm

$$\|u\|_{BC(I,\mathbb{R})} := \sup_{t\geq t_0-\sigma} |u(t)|.$$

Suppose that Ω is nonempty subset of $BC(I,\mathbb{R})$.

Definition 2.2 ([13])**.** *We say that the solutions of Equation* (1.1) *are uniformly locally attractive if there exists a ball $B(u_0,r_0)$ in the space $BC(I,\mathbb{R})$ such that for each $\epsilon > 0$ there exists $T > 0$ such that for arbitrary solutions $u,v \in B(u_0,r_0)\cap\Omega$ we have*

$$|u(t)-v(t)| \leq \epsilon \qquad t \geq T. \tag{2.1}$$

Theorem 2.3 (Schauder fixed point theorem)**.** *Let K be a nonempty convex subset of a Hausdorff topological vector space X. Let T be a continuous mapping of K into itself such that $T(K)$ is contained in a compact subset of K. Then T has a fixed point in K.*

Now, we mention some results and facts about measure of noncompactness.

Let E be a Banach space. We denote by $\overline{X}$ and Conv X the closure and the convex closure of X as a subset of E, respectively. Further, we denote by $\mathfrak{M}_E$ the family of all nonempty bounded subsets of E and $\mathfrak{R}_E$ its subfamily consisting of all relatively compact sets.

Definition 2.4 ([12])**.** *A mapping $\mu : \mathfrak{M}_E \to \mathbb{R}^+$ is said to be a measure of noncompactness in E if satisfies the following conditions:*

(a) The family $\ker\mu = \{X \in \mathfrak{M}_E : \mu(X) = 0\}$ is nonempty and $\ker\mu \subset \mathfrak{R}_E$.
(b) $X \subset Y$ implies $\mu(X) \leq \mu(Y)$.
(c) $\mu(\overline{X}) = \mu(X)$.
(d) $\mu(Conv\,(X)) = \mu(X)$.
(e) For all $\lambda \in [0,1]$,

$$\mu(\lambda X + (1-\lambda)Y) \leq \lambda\mu(X) + (1-\lambda)\mu(Y).$$

(f) If $(X_n)_{n\in\mathbb{N}}$ *is a sequence of closed sets from* $\mathfrak{M}_E$ *such that*

$$X_{n+1} \subset X_n \quad \text{for all} \quad n = 1, 2, 3... \quad \text{and} \quad \lim_{n\to\infty} \mu(X_n) = 0,$$

then

$$X_\infty = \bigcap_{n=1}^{\infty} X_n \neq \emptyset.$$

The family $\ker \mu$ *described in* (a) *is said to be the kernel of the measure of noncompactness* μ.

The following result will be the key tool in the proof of one of our main results.

Theorem 2.5. [4, Theorem 2.2] *Let* C *be a nonempty, bounded, closed, and convex subset of a Banach space* E *and let* $T : C \to C$ *be a continuous function satisfying*

$$\mu(T(W)) \leq \phi(\mu(W))$$

for each $W \subset C$*, where* μ *is an arbitrary measure of noncompactness and* $\phi : [0, \infty) \to [0, \infty)$ *is a monotone increasing (not necessarily continuous) function with*

$$\lim_{n\to\infty} \phi^n(t) = 0 \quad \text{for all} \quad t \geq 0.$$

Then T *has at least one fixed point in* C.

3. Existence of attractive solutions

In this section we investigate attractive solutions of the problem

$$u(t) = \begin{cases} \phi(t_0) + \displaystyle\int_{t_0}^{t} a(t-s) f(s, u_s)\, ds & t > t_0, \\ \phi(t) & t_0 - \sigma \leq t \leq t_0, \end{cases} \tag{3.1}$$

where $a \in L^1_{loc}(\mathbb{R}_+)$ is an scalar kernel, $\phi \in C([t_0 - \sigma, t_0], \mathbb{R})$, $f : [t_0, \infty) \times C([-\sigma, 0], \mathbb{R}) \to \mathbb{R}$ and $u_s : [-\sigma, 0] \to \mathbb{R}$ is given by $u_s(\tau) := u(s + \tau)$ for all $\tau \in [-\sigma, 0]$ and $s > t_0$ where $u : [t_0 - \sigma, \infty) \to \mathbb{R}$.

We need the following assumptions

- (H1) (a) For each $\psi \in C([t_0 - \sigma, \infty), \mathbb{R})$ the function $\Psi : [t_0, \infty) \to \mathbb{R}$ defined by $\Psi(t) = f(t, \psi_t)$ is Lebesgue measurable.
  - (b) For each $t > t_0$ the function $F : C([-\sigma, 0], \mathbb{R}) \to \mathbb{R}$ defined by $F(\psi) = f(t, \psi)$ is continuous.
- (H2) There exists an strictly decreasing function $\mathcal{H} : [0, \infty) \to [0, \infty)$ such that for all $\psi \in C([t_0 - \sigma, \infty), \mathbb{R})$,

$$\left| \phi(t_0) + \int_{t_0}^{t} a(t-s) f(s, \psi_s)\, ds \right| \leq \mathcal{H}(t - t_0) \quad \text{for all} \quad t \geq t_0,$$

  and

$$\lim_{t\to\infty} \mathcal{H}(t) = 0.$$

- (H3) There exists $0 < \beta < 1$ such that
  - (a) $f \in L^{\frac{1}{\beta}}([t_0, \infty) \times C([-\sigma, 0]; \mathbb{R}), \mathbb{R})$,
  - (b) $a \in L^{\frac{1}{\beta}}([0, \infty), \mathbb{R})$ satisfies

$$\lim_{h\to 0} \int_0^t |a(\tau + h) - a(\tau)|^{\frac{1}{1-\beta}}\, d\tau = 0, \quad t > 0.$$

Next, we present the main theorem of this section.

Theorem 3.1. *Suppose that the hypothesis* $(H1)$*-*$(H3)$ *hold. Then the Equation* (1.1) *has at least one attractive solution in* $C([t_0-\sigma,\infty),\mathbb{R})$.

Proof. Define the set

$$S := \{u \in C([t_0-\sigma,\infty),\mathbb{R}) : |u(t)| \le \mathcal{H}(t-t_0) \text{ for all } t \ge t_0\}.$$

It can be easily verified that S is nonempty and convex. We show that S is bounded. Indeed, using (H2) we have

$$\sup_{t\ge t_0-\sigma}|u(t)| = \sup_{s\ge -\sigma}|u(s+t_0)| \le \begin{cases} \mathcal{H}(s) & \text{if } s+t_0 \ge t_0, \\ \sup\limits_{-\sigma\le s\le 0}|u(s+t_0)| =: M & \text{if } -\sigma \le s \le 0 \end{cases} \le \max\{\mathcal{H}(0), M\},$$

proving the claim. We define

$$(\mathcal{G}u)(t) = \begin{cases} \phi(t_0) + \displaystyle\int_{t_0}^{t} a(t-s) f(s,u_s)\, ds & t > t_0, \\ \phi(t) & t_0-\sigma \le t \le t_0 \end{cases} \tag{3.2}$$

for every $u \in C([t_0-\sigma,\infty),\mathbb{R})$. We have that $\mathcal{G}(S) \subset S$ by $(H2)$. Now, we prove that $\mathcal{G}$ is continuous. Let $(u^n)_{n\in\mathbb{N}}$ be a sequence in S such that $\lim_{n\to\infty} u^n = u$ uniformly on compact subsets of $[t_0-\sigma,\infty)$. By $(H2)$ we have that $\mathcal{H}$ vanish at the infinity and is strictly decreasing, therefore given $\epsilon > 0$ there exists $T > t_0$ such that

$$\mathcal{H}(t-t_0) < \epsilon \quad \text{for all} \quad t > T. \tag{3.3}$$

On the other hand, by $(H1)$ item (b), there exists $\delta > 0$ such that

$$\|\psi - \psi_0\|_{C([-\sigma,0],\mathbb{R})} < \delta \quad \text{implies} \quad |f(t,\psi) - f(t,\psi_0)| < \epsilon. \tag{3.4}$$

For this $\delta > 0$ there exists $N \in \mathbb{N}$ such that $|u^n(t) - u(t)| < \delta$ for all $n > N$ and for all t uniformly on compact subsets of $[t_0-\sigma,\infty)$. Now, note that

$$\|u_s\|_{C([-\sigma,0],\mathbb{R})} := \sup_{-\sigma\le t\le 0}|u_s(t)| = \sup_{-\sigma\le t\le 0}|u(s+t)| = \sup_{s-\sigma\le\tau\le s}|u(\tau)| \tag{3.5}$$

for all $s > t_0$. Therefore $\|u^n_s - u_s\|_{C([-\sigma,0],\mathbb{R})} = \sup_{s-\sigma\le\tau\le s}|u^n(\tau) - u(\tau)| < \delta$ for all $s > t_0$. It follows from (3.4) that

$$|f(s,u^n_s) - f(s,u_s)| < \epsilon, \tag{3.6}$$

for all $s > t_0$. Now, we divide the proof of continuity in three intervals. First, we observe that for $t_0 < t \le T$ we have by Hölder inequality, (H3) and (3.6) that the following estimate holds

$$\begin{aligned} |\mathcal{G}(u^n)(t) - \mathcal{G}(u)(t)| &\le \int_{t_0}^{t} |a(t-s)||f(s,u^n_s) - f(s,u_s)|\, ds \\ &\le \left(\int_{t_0}^{t} |a(t-s)|^{\frac{1}{1-\beta}}\, ds\right)^{1-\beta} \left(\int_{t_0}^{t} |f(s,u^n_s) - f(s,u_s)|^{\frac{1}{\beta}}\, ds\right)^{\beta} \\ &\le \left(\int_{t_0}^{T} |a(t-s)|^{\frac{1}{1-\beta}}\, ds\right)^{1-\beta} \left(\int_{t_0}^{T} |f(s,u^n_s) - f(s,u_s)|^{\frac{1}{\beta}}\, ds\right)^{\beta} \\ &\le \left(\int_{0}^{T-t_0} |a(\tau)|^{\frac{1}{1-\beta}}\, d\tau\right)^{1-\beta} \left(\int_{t_0}^{T} \epsilon^{\frac{1}{\beta}}\, ds\right)^{\beta} \\ &< \left(\int_{0}^{T-t_0} |a(\tau)|^{\frac{1}{1-\beta}}\, d\tau\right)^{1-\beta} (T-t_0)^{\beta}\epsilon. \end{aligned}$$

Second, for $T < t$ we have by (H2), the S-invariance of $\mathcal{G}$ and (3.3) the following estimate

$$|\mathcal{G}(u^n)(t) - \mathcal{G}(u)(t)| \leq |\mathcal{G}(u^n)(t)| + |\mathcal{G}(u)(t)| \leq 2\mathcal{H}(t - t_0) < 2\epsilon.$$

Finally, for the interval $t_0 - \sigma \leq t \leq t_0$ we have by definition that $|\mathcal{G}(u^n)(t) - \mathcal{G}(u)(t)| = 0$. It follows from $(A1)$ that $\lim_{n\to\infty} \mathcal{G}(u^n) = \mathcal{G}(u)$ uniformly on compact subsets of $[t_0 - \sigma, \infty)$. This proves that $\mathcal{G}$ is continuous.

Let $\epsilon > 0, u \in S$ be given and $T > t_0$ such that (3.3) holds. We claim that $\mathcal{G}(S)$ is equicontinuous on $[t_0, T]$. Indeed, first we consider the case $t_1, t_2 \in (t_0, T]$. Without loss of generality, we can assume $t_1 < t_2$. Then by Hölder inequality and $(H3)$ we obtain

$$\begin{aligned}
|\mathcal{G}(u)(t_2) - \mathcal{G}(u)(t_1)| &= \left| \int_{t_0}^{t_2} a(t_2 - s) f(s, u_s)\, ds - \int_{t_0}^{t_1} a(t_1 - s) f(s, u_s)\, ds \right| \\
&\leq \int_{t_0}^{t_1} |a(t_2 - s) - a(t_1 - s)||f(s, u_s)|\, ds + \int_{t_1}^{t_2} |a(t_2 - s)||f(s, u_s)|\, ds \\
&\leq \left(\int_{t_0}^{t_1} |a(t_2 - s) - a(t_1 - s)|^{\frac{1}{1-\beta}}\, ds \right)^{1-\beta} \left(\int_{t_0}^{T} |f(s, u_s)|^{\frac{1}{\beta}}\, ds \right)^{\beta} \\
&+ \left(\int_{t_1}^{t_2} |a(t_2 - s)|^{\frac{1}{1-\beta}}\, ds \right)^{1-\beta} \left(\int_{t_1}^{T} |f(s, u_s)|^{\frac{1}{\beta}}\, ds \right)^{\beta} \\
&= \left(\int_{0}^{t_1 - t_0} |a((t_2 - t_1) + \tau) - a(\tau)|^{\frac{1}{1-\beta}}\, d\tau \right)^{1-\beta} \left(\int_{t_0}^{T} |f(s, u_s)|^{\frac{1}{\beta}}\, ds \right)^{\beta} \\
&+ \left(\int_{0}^{t_2 - t_1} |a(\tau)|^{\frac{1}{1-\beta}}\, d\tau \right)^{1-\beta} \left(\int_{t_1}^{T} |f(s, u_s)|^{\frac{1}{\beta}}\, ds \right)^{\beta} \\
&\leq \left(\int_{0}^{T - t_0} |a((t_2 - t_1) + \tau) - a(\tau)|^{\frac{1}{1-\beta}}\, d\tau \right)^{1-\beta} \left(\int_{t_0}^{T} |f(s, u_s)|^{\frac{1}{\beta}}\, ds \right)^{\beta} \\
&+ \left(\int_{0}^{t_2 - t_1} |a(\tau)|^{\frac{1}{1-\beta}}\, d\tau \right)^{1-\beta} \left(\int_{t_1}^{T} |f(s, u_s)|^{\frac{1}{\beta}}\, ds \right)^{\beta} \to 0 \quad \text{as} \quad t_2 \to t_1.
\end{aligned}$$

Secondly, suppose that $T < t_1, t_2$. Then by (H2), the S-invariance of $\mathcal{G}$ and (3.3) we obtain

$$|\mathcal{G}(u)(t_2) - \mathcal{G}(u)(t_1)| \leq |\mathcal{G}(u)(t_2)| + |\mathcal{G}(u)(t_1)| \leq \mathcal{H}(t_2 - t_0) + \mathcal{H}(t_1 - t_0) < 2\epsilon.$$

Finally, suppose that $t_0 < t_1 < T < t_2$. Then, by the preceding cases, we have

$$|\mathcal{G}(u)(t_2) - \mathcal{G}(u)(t_1)| \leq |\mathcal{G}(u)(t_2) - \mathcal{G}(u)(T)| + |\mathcal{G}(u)(T) - \mathcal{G}(u)(t_1)| \to 0 \quad (t_2 \to t_1).$$

Consequently, we conclude that $\mathcal{G}(S)$ is equicontinuous on each compact interval $[t_0, T]$ for all $T > t_0$, proving the claim. It follows from the Arzela-Ascoli theorem that $\mathcal{G}(S)$ is relatively compact set in $C([t_0, T], \mathbb{R})$ for all $T > t_0$.

Now, we show that $\mathcal{G}(S)$ is uniformly bounded. Indeed, by definition of the set S and since $\mathcal{H}$ is strictly decreasing, we have

$$\sup_{u \in \mathcal{G}(S)} \sup_{t > T} |u(t)| \leq \mathcal{H}(T - t_0),$$

therefore from (H2) we obtain

$$\lim_{T\to\infty} \sup_{u \in \mathcal{G}(S)} \sup_{t > T} |u(t)| = 0.$$

It follows from $(A2)$ that the family $\mathcal{G}(S)$ is relatively compact in $C([t_0 - \sigma, \infty), \mathbb{R})$. By Schauder fixed point theorem we have that the operator $\mathcal{G}$ has a fixed point in S. Hence the Equation

(1.1) has at least one solution $u_0 \in S$. Choosing $b_0 := \sup_{t\in[t_0-\sigma,t_0]} |u_0(t)|$ we have $|\phi(t)| \leq b_0,\ t \in [t_0-\sigma, t_0]$ and from

$$|u_0(t)| \leq \mathcal{H}(t-t_0) \qquad t \geq t_0,$$

it follows that the solution of (1.1) is attractive.

□

As a particular case, we recover [18, Theorem 3.2].

Corollary 3.2. *Suppose (H1) and the following*

(H2)' *There exists* $\gamma_1 > 0$ *such that*

$$\left|\phi(t_0) + \int_{t_0}^{t} \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} f(s,u_s)\, ds\right| \leq (t-t_0)^{-\gamma_1} \quad \text{for all} \quad t \geq t_0,$$

(H3)' *There exists* $0 < \beta < \alpha$ *such that* $f \in L^{\frac{1}{\beta}}([t_0,\infty) \times C([-\sigma,0];\mathbb{R}),\mathbb{R})$.

Then the Equation

$$u(t) = \begin{cases} \phi(t_0) + \displaystyle\int_{t_0}^{t} \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} f(s,u_s)\, ds & t > t_0, \\ \phi(t) & t_0 - \sigma \leq t \leq t_0, \end{cases} \tag{3.7}$$

has at least one attractive solution in $C([t_0-\sigma,\infty),\mathbb{R})$ *for each* $0 < \alpha < 1$.

Proof. Let $a(t) = \frac{t^{\alpha-1}}{\Gamma(\alpha)}$ in Theorem 3.1. In view of $(H2)'$ we can choose $\mathcal{H}(t) = t^{-\gamma_1}$ and therefore (H2) in Theorem 3.1 holds. From $(H3)'$ there exists $0 < \beta < \alpha$ such that

$$\begin{aligned}\frac{1}{\Gamma(\alpha)^{\frac{1}{1-\beta}}} \int_0^t [(\tau+h)^{\alpha-1} - \tau^{\alpha-1}]^{1/1-\beta} d\tau &\leq \frac{1}{\Gamma(\alpha)^{\frac{1}{1-\beta}}} \int_0^t [(\tau+h)^{\frac{\alpha-1}{1-\beta}} - \tau^{\frac{\alpha-1}{1-\beta}}] d\tau \\ &= \frac{1}{\Gamma(\alpha)^{\frac{1}{1-\beta}}} \frac{1-\beta}{\alpha-\beta} [(t+h)^{\frac{\alpha-\beta}{1-\beta}} - h^{\frac{\alpha-\beta}{1-\beta}} - t^{\frac{\alpha-\beta}{1-\beta}}] \to 0,\end{aligned}$$

as $h \to 0$, where we have used the inequality $(p-q)^r \leq p^r - q^r$, $p > q > 0, r > 0$. This shows (H3) and the conclusion follows from Theorem 3.1.

□

Remark 3.3. *In the same way as in the above corollary, we recover* [24, Theorem 3.2]. *We only need to choose* $a(t) = \displaystyle\sum_{i=1}^{m} \frac{t^{\alpha-\alpha_i-1}}{\Gamma(\alpha-\alpha_i)}$ *where* $0 < \alpha_i < \alpha,\ \alpha > 0$ *in order to have* $(H2)$ *and* $0 < \beta < \min_{0\leq i\leq m} \{\alpha - \alpha_i\}$ *in order to obtain* $(H3)$.

4. Uniform local attractivity

Let $B \subset BC(I,\mathbb{R})$ be a bounded subset and $T > t_0 - \sigma$ given, where $I := [t_0-\sigma,\infty)$. For $\epsilon > 0$ and $u \in B$ we denote

$$\begin{aligned}\omega_{t_0-\sigma}^{T}(u,\epsilon) &:= \sup\{|u(t)-u(s)| : t,s \in [t_0-\sigma,T],\ |t-s| \leq \epsilon\}, \\ \omega_{t_0-\sigma}^{T}(B,\epsilon) &:= \sup\{\omega_{t_0-\sigma}^{T}(u,\epsilon) : u \in B\}, \\ \omega_{t_0-\sigma}^{T}(B) &:= \lim_{\epsilon\to 0} \omega_{t_0-\sigma}^{T}(B,\epsilon), \\ \omega_{t_0-\sigma}(B) &:= \lim_{T\to\infty} \omega_{t_0-\sigma}^{T}(B).\end{aligned}$$

If $t \geq t_0 - \sigma$ is a fixed number, then let us denote

$$B(t) := \{u(t) : u \in B\},$$

and

$$\text{diam } B(t) := \sup_{u,v \in B} |u(t) - v(t)|.$$

Finally, we consider the mapping μ defined on the family $\mathfrak{M}_{BC(I,\mathbb{R})}$ by

$$\mu(B) := \omega_{t_0-\sigma}(B) + \limsup_{t\to\infty} \text{diam } B(t).$$

It can be shown that μ is a measure of noncompactness on $BC(I, \mathbb{R})$ in the same way that in [12].

In this section we will need the following hypothesis.

(H4) $f : [t_0-\sigma, \infty)\times C([t_0-\sigma, t_0], \mathbb{R}) \to \mathbb{R}$ is continuous and there exists $h : [t_0-\sigma, \infty) \to [0, \infty)$ continuous such that

$$|f(t, u) - f(t, v)| \leq h(t)\mathcal{H}(\|u - v\|) \quad \text{for all} \quad t \geq t_0 - \sigma,\ u, v \in C([t_0 - \sigma, t_0], \mathbb{R}),$$

where $\mathcal{H} : [0, \infty) \to [0, \infty)$ is superadditive, that is,

$$\mathcal{H}(a) + \mathcal{H}(b) \leq \mathcal{H}(a + b) \quad a, b \geq 0.$$

(H5) The following properties hold

$$A := \sup_{t\geq t_0} \int_{t_0}^{t} |a(t - s)||h(s)|ds < 1;$$

$$B := \sup_{t\geq t_0} \int_{t_0}^{t} |a(t - s)||f(s, 0)|\, ds < \infty;$$

and

$$\lim_{n\to\infty} A^n\mathcal{H}^n(t) = 0, \quad t > 0.$$

(H6) There exists a positive solution r_0 of the inequality

$$\left[\sup_{t\in[t_0-\sigma,t_0]} \phi(t)\right] + A\mathcal{H}(r) + B \leq r.$$

Remark 4.1. *Note that $\mathcal{H}$ is an nondecreasing function on $[0, \infty)$. Indeed, given $a, b \geq 0$ with $a \leq b$ we obtain that*

$$\mathcal{H}(b) \geq \mathcal{H}(b - a) + \mathcal{H}(a) \geq \mathcal{H}(a).$$

The following is the main result of this section.

Theorem 4.2. *Suppose that $(H3)$-$(H6)$ hold. Then the Equation* (1.1) *has at least one solution in $BC(I, \mathbb{R})$. Moreover, these solutions are uniform locally attractive.*

Proof. We define

$$(\mathcal{G}u)(t) = \begin{cases} \phi(t_0) + \displaystyle\int_{t_0}^{t} a(t - s)f(s, u_s)\, ds & t > t_0, \\ \phi(t) & t_0 - \sigma \leq t \leq t_0, \end{cases} \tag{4.1}$$

for every $u \in BC(I, \mathbb{R})$.

We first prove the existence of at least one solution. In order to do this, we proceed to verify the hypothesis of Theorem 2.5 dividing the proof in four steps.

Step 1. $BC(I,\mathbb{R})$ is invariant under $\mathcal{G}$. Indeed, for any $u \in BC(I,\mathbb{R})$ and $t > t_0$ we get from (H4) and (H5)

$$\begin{aligned}
|(\mathcal{G}(u))(t)| &\leq |\phi(t_0)| + \int_{t_0}^{t} |a(t-s)||f(s,u_s) - f(s,0)|\,ds + \int_{t_0}^{t} |a(t-s)||f(s,0)|\,ds \\
&\leq |\phi(t_0)| + \int_{t_0}^{t} |a(t-s)|h(s)\mathcal{H}(\|u_s\|_{C([-\sigma,0],\mathbb{R})})\,ds + \int_{t_0}^{t} |a(t-s)||f(s,0)|\,ds \\
&\leq |\phi(t_0)| + \int_{t_0}^{t} |a(t-s)|h(s)\mathcal{H}(\|u\|_{BC(I,\mathbb{R})})\,ds + \int_{t_0}^{t} |a(t-s)||f(s,0)|\,ds \\
&\leq |\phi(t_0)| + A\mathcal{H}(\|u\|_{BC(I,\mathbb{R})}) + B,
\end{aligned}$$

where in the third inequality we have used (3.5). This shows that $\mathcal{G}(u)$ is bounded in $[t_0,\infty)$. Since $\phi \in C([t_0-\sigma,t_0],\mathbb{R})$ we conclude that $\mathcal{G}(u) \in BC(I,\mathbb{R})$.

Let $r_0 > 0$ be the real number given in $(H6)$. We set

$$C := \{u \in BC(I,\mathbb{R}) : \|u\|_{BC(I,\mathbb{R})} \leq r_0\}.$$

Step 2. We show that $\mathcal{G}(C) \subset C$. Indeed, let $u \in C$ and $t > t_0$. Then by step 1 and $(H6)$, we obtain

$$|\mathcal{G}(u)(t)| \leq |\phi(t_0)| + A\mathcal{H}(\|u\|_{BC(I,\mathbb{R})}) + B \leq \sup_{t_0-\sigma\leq t\leq t_0} |\phi(t)| + A\mathcal{H}(r_0) + B \leq r_0,$$

where we used Remark 4.1. On the other hand, for $t_0 - \sigma \leq t \leq t_0$ we have

$$|\mathcal{G}(u)(t)| = |\phi(t)| \leq \sup_{t_0-\sigma\leq t\leq t_0} |\phi(t)| \leq \sup_{t_0-\sigma\leq t\leq t_0} |\phi(t)| + A\mathcal{H}(r_0) + B < r_0.$$

Hence $\mathcal{G}(C) \subset C$.

Step 3. We prove that

$$\limsup_{t\to\infty} \text{diam } (\mathcal{G}(W))(t) \leq A\mathcal{H}(\limsup_{t\to\infty} \text{diam } W(t)),$$

for any $W \subset C$. In fact, let $u, v \in W$ and $t > t_0$. Then by (H4), (H6) and (3.5) we obtain

$$\begin{aligned}
|(\mathcal{G}(u))(t) - (\mathcal{G}(v))(t)| &\leq \int_{t_0}^{t} |a(t-s)||f(s,u_s) - f(s,v_s)|\,ds \leq \int_{t_0}^{t} |a(t-s)|h(s)\mathcal{H}(\|u_s - v_s\|)\,ds \\
&\leq A\mathcal{H}(\|u-v\|).
\end{aligned}$$

Since $\mathcal{H}$ is increasing, the inequality $\|u-v\| \leq \sup_{u,v\in W}\|u-v\|$ implies $\mathcal{H}(\|u-v\|) \leq \mathcal{H}(\sup_{u,v\in W}\|u-v\|)$. It follows that

$$\text{diam } (\mathcal{G}(W))(t) \leq A\mathcal{H}(\|u-v\|) \leq A\mathcal{H}(\sup_{u,v\in W}\|u-v\|) \leq A\mathcal{H}(\text{diam } W(t)).$$

Hence

$$\limsup_{t\to\infty} \text{diam } (\mathcal{G}(W))(t) \leq A\mathcal{H}(\limsup_{t\to\infty} \text{diam } W(t)).$$

Since for $t_0 - \sigma \leq t \leq t_0$ we have $|(\mathcal{G}(u))(t) - (\mathcal{G}(v))(t)| = 0$, and the claim follows.

Step 4. We prove that $\mu(\mathcal{G}(W)) \leq \varphi(\mu(W))$ where $\varphi(t) := A\mathcal{H}(t)$, $t > 0$ and μ is an arbitrary measure of noncompactness.

Indeed, note that by $(H5)$ we have $\lim_{n\to\infty}\varphi^n(t)=0$ for all $t>0$. Let $u\in W\subset C$, $t_1,t_2\in[t_0-\sigma,t_0]$, $t_1<t_2$, $T>t_0$, $\epsilon>0$ and $|t_1-t_2|\le\epsilon$. Then

$$\begin{aligned}
|\mathcal{G}(u)(t_2)-\mathcal{G}(u)(t_1)| &\le \left|\int_{t_0}^{t_2} a(t_2-s)f(s,u_s)\,ds-\int_{t_0}^{t_1} a(t_1-s)f(s,u_s)\,ds\right|\\
&\le \int_{t_0}^{t_1}|a(t_2-s)-a(t_1-s)||f(s,u_s)|\,ds+\int_{t_1}^{t_2}|a(t_2-s)||f(s,u_s)|\,ds\\
&\le \int_{t_0}^{t_1}|a(t_2-s)-a(t_1-s)||f(s,u_s)|\,ds\\
&+\int_{t_1}^{t_2}|a(t_2-s)||f(s,u_s)-f(s,0)|\,ds+\int_{t_1}^{t_2}|a(t_2-s)|f(s,0)|\,ds\\
&\le \int_{t_0}^{t_1}|a(t_2-s)-a(t_1-s)||f(s,u_s)|\,ds\\
&+\int_{t_1}^{t_2}|a(t_2-s)|\left[h(s)\mathcal{H}(\|u_s\|)+|f(s,0)|\right]\,ds\\
&\le \int_{t_0}^{t_1}|a(t_2-s)-a(t_1-s)||f(s,u_s)|\,ds\\
&+\int_{t_1}^{t_2}|a(t_2-s)|\left[h(s)\mathcal{H}(r_0)+|f(s,0)|\right]\,ds\\
&\le \omega_1^T(f,\epsilon)+\omega_2^T(f,\epsilon)+A\mathcal{H}(\omega_{t_0-\sigma}^T(u,\epsilon));
\end{aligned}$$

where

$$\omega_1^T(f,\epsilon):=\sup\left\{\int_{t_0}^{t_1}|a(t_2-s)-a(t_1-s)||f(s,u_s)|\,ds:\ t_1,t_2\in[t_0-\sigma,T],|t_2-t_1|<\epsilon,\|u\|\le r_0\right\},$$

and

$$\omega_2^T(f,\epsilon):=\sup\left\{\int_{t_1}^{t_2}|a(t_2-s)|\left[h(s)\mathcal{H}(r_0)+|f(s,0)|\right]\,ds:\ t_1,t_2\in[t_0-\sigma,T],|t_2-t_1|<\epsilon\right\}.$$

Thus

$$\omega_{t_0-\sigma}^T(\mathcal{G}(u),\epsilon)\le\omega_1^T(f,\epsilon)+\omega_2^T(f,\epsilon)+A\mathcal{H}(\omega_{t_0-\sigma}^T(u,\epsilon)).$$

Then

$$\omega_{t_0-\sigma}^T(\mathcal{G}(W),\epsilon)\le\omega_1^T(f,\epsilon)+\omega_2^T(f,\epsilon)+A\mathcal{H}(\omega_{t_0-\sigma}^T(W,\epsilon)).$$

It follows from $(H3)$ that

$$\omega_{t_0-\sigma}^T(\mathcal{G}(W))=\lim_{\epsilon\to0}\omega_{t_0-\sigma}^T(\mathcal{G}(W),\epsilon)\le A\mathcal{H}(\omega_{t_0-\sigma}^T(W)).$$

Consequently

$$\omega_{t_0-\sigma}(\mathcal{G}(W))=\lim_{T\to\infty}\omega_{t_0-\sigma}^T(\mathcal{G}(W))\le A\mathcal{H}(\omega_{t_0-\sigma}(W)).$$

By the superadditivity of $\mathcal{H}$

$$\mu(\mathcal{G}(W)):=\omega_{t_0-\sigma}(\mathcal{G}(W))+\limsup_{t\to\infty}\operatorname{diam}\ (\mathcal{G}(W))(t)\le A\mathcal{H}(\omega_{t_0-\sigma}(W))+A\mathcal{H}(\limsup_{t\to\infty}\operatorname{diam}W(t))$$

$$\le A\mathcal{H}\left[\omega_{t_0-\sigma}(W)+\limsup_{t\to\infty}\operatorname{diam}W(t)\right]=\varphi(\mu(W)). \tag{4.2}$$

From steps 1-4 we have that all the hypothesis of Theorem 2.5 are satisfied and hence $\mathcal{G}$ has a fixed point on $BC(I,\mathbb{R})$.

Finally, we prove that all solutions of Equation (1.1) are uniformly locally attractive in the sense of Definition 2.2. Indeed, let $B_{r_0} = \{u \in BC(I,\mathbb{R}) : \|u\| \le r_0\}$, $B^1_{r_0} := \text{Conv}\, \mathcal{G}(B_{r_0})$, $B^2_{r_0} := \text{Conv}\, \mathcal{G}(B^1_{r_0})$ and so on. Then $B^1_{r_0} \subset B_{r_0}$, $B^{n+1}_{r_0} \subset B^n_{r_0}$ for $n = 1, 2, \dots$. It is clear that these sets are nonempty, closed and convex. Using the inequality (4.2) and by an inductive argument we obtain that

$$\mu(B^n_{r_0}) \le A^n \mathcal{H}^n(\mu(B_{r_0})) \quad n = 1, 2, 3, \dots .$$

From $(H5)$ we conclude that

$$\lim_{n\to\infty} \mu(B^n_{r_0}) = 0.$$

Let $B = \cap_{n=1}^{\infty} B^n_{r_0}$. As in [24, Theorem 4.8], we get that the set B is nonempty, bounded, closed, convex and invariant under $\mathcal{G}$. Then

$$\lim_{t\to\infty} \sup_{u,v\in B} |u(t) - v(t)| \le \omega_{t_0-\sigma}(B) + \lim_{t\to\infty} \text{diam}\, B(t) = \mu(B) \le \lim_{k\to\infty} \mu(B^k_{r_0}) = 0.$$

We conclude that all solutions are uniformly locally attractive. □

5. Applications

Example 5.1. *We consider the following integral equation*

$$u(t) = \begin{cases} \displaystyle\int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} e^{-\delta(t-s)} e^{-\gamma s} e^{-\sin u(s-1)}\, ds & t > 0, \\ te^{-t} & -1 \le t \le 0. \end{cases} \tag{5.1}$$

where $\delta > \gamma > 0$ and $\alpha > 0$. Choosing $\phi(t) := te^{-t}$ for $t \in [-1,0]$, $a(t) := \frac{t^{\alpha-1}}{\Gamma(\alpha)} e^{-\delta t}$, $t \ge 0$, and $f(s,\varphi) := e^{-\gamma s} e^{-\sin \varphi(-1)}$ where $\varphi \in C([-1,0],\mathbb{R})$ we have that (5.1) is in the form of (3.1) with $t_0 = 0$ and $\sigma = 1$. Observe that $u_s : [-1,0] \to \mathbb{R}$, $u_s(-1) = u(s-1)$ and therefore $f(s,u_s) = e^{-\gamma s} e^{-\sin u(s-1)}$. We will verify the conditions (H1)-(H3).

$(H1)$ *(a) For every $u \in C([-1,\infty),\mathbb{R})$ we have the function*

$$t \mapsto f(t,u_t) = e^{-\gamma t} e^{-\sin u(t-1)}$$

is Lebesgue measurable since it is a continuous function for all $t \ge 0$.

(b) Let $\psi \in C([-1,0],\mathbb{R})$. We see that the function

$$\begin{aligned} F : C([-1,0],\mathbb{R}) &\to \mathbb{R} \\ \psi &\mapsto f(t,\psi) \end{aligned}$$

is continuous. Indeed, let $(\psi^n)_{n\in\mathbb{N}}$ be a sequence in $C([-1,0],\mathbb{R})$ such that $\|\psi^n - \psi\|_{C([-1,0],\mathbb{R})} \to 0$ as $n \to \infty$. Applying the mean value theorem we obtain

$$\begin{aligned} |f(t,\psi^n) - f(t,\psi)| &= e^{-\gamma t} |e^{-\sin \psi^n(-1)} - e^{-\sin \psi(-1)}| \le e^{-\gamma t} e |\psi^n(-1) - \psi(-1)| \\ &\le e^{1-\gamma t} \|\psi^n - \psi\| \to 0 \quad (n \to \infty). \end{aligned}$$

$(H2)$ *Let $\mathcal{H} : [0,\infty) \to [0,\infty)$ defined by*

$$\mathcal{H}(t) := \frac{e^{1-\gamma t}}{(\delta-\gamma)^{\alpha}}, \quad t \ge 0.$$

Then $\mathcal{H}$ vanish at the infinity and for $t \geq 0$ and we have

$$\left|\int_0^t a(t-s)e^{-\gamma s}e^{-\sin u(s-1)}\,ds\right| = e\int_0^t a(t-s)e^{-\gamma s}\,ds = e\int_0^t a(\tau)e^{-\gamma(t-\tau)}\,d\tau$$
$$\leq e^{1-\gamma t}\int_0^\infty e^{\gamma\tau}a(\tau)\,d\tau = \frac{e^{1-\gamma t}}{(\delta-\gamma)^\alpha},$$

where in the last equality we have used [21, Formula 3.381(4)].

$(H3)$ *Let $1-\alpha<\beta<1$. Using again* [21, Formula 3.381(4)] *we obtain*

$$\int_0^\infty |a(s)|^{1/\beta}ds = \int_0^\infty \frac{s^{\frac{\alpha-1}{\beta}}}{\Gamma(\alpha)^{1/\beta}}e^{-\frac{\delta}{\beta}s}ds = \frac{\Gamma(\frac{\alpha-1+\beta}{\beta})}{\Gamma(\alpha)^{1/\beta}}\left(\frac{\beta}{\delta}\right)^{\frac{\alpha-1+\beta}{\beta}}. \tag{5.2}$$

Moreover,

$$\int_0^\infty |f(s,\psi)|^{1/\beta}ds \leq \int_0^\infty e^{-\frac{\gamma}{\beta}s}e^{-\frac{\sin\psi(-1)}{\beta}}ds \leq \frac{e\beta}{\gamma}$$

for each $\psi\in C([-1,0],\mathbb{R})$. Finally, from the continuity of $a(t)$ it is clear that

$$\lim_{h\to 0}\int_0^t |a(s+h)-a(s)|^{\frac{1}{1-\beta}}\,ds = 0. \tag{5.3}$$

Hence, all conditions of Theorem 3.1 are satisfied and we can conclude that the Equation (5.1) *has at least one attractive solution in $C([-1,\infty),\mathbb{R})$ whenever $\delta>\gamma>0$ and $\alpha>0$.*

Example 5.2. *We consider the problem*

$$u(t) = \begin{cases} \displaystyle\int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}e^{-\delta(t-s)}\frac{s^2}{1+s^4}\ln(1+|u(s-1/2)|)\,ds & t>0,\\ te^{\frac{t}{\pi}} & -\frac{\pi}{2}\leq t\leq 0,\end{cases} \tag{5.4}$$

where $\delta>1$ and $\alpha>0$. Here $\phi(t):=te^{\frac{t}{\pi}}$ for $t\in[-\frac{\pi}{2},0]$, $a(t):=\frac{t^{\alpha-1}}{\Gamma(\alpha)}e^{-\delta t}$, $t\geq 0$, and $f(s,\psi):=\dfrac{s^2}{1+s^4}\ln(1+|\psi(-\pi/2)|)$ for $\psi\in C([-\frac{\pi}{2},0],\mathbb{R})$. Equation (5.4) *has the form of the problem* (3.1) *with $t_0=0$ and $\sigma=\frac{\pi}{2}$. We will verify conditions $(H3)-(H6)$.*

$(H3)$ *Let $1-\alpha<\beta<1$. From* [21, Formula 3.251 (11)] *we obtain that*

$$\int_0^\infty |f(s,\psi)|^{1/\beta}ds = \ln(1+|\psi(-\pi/2)|)^{1/\beta}\int_0^\infty \frac{s^{2/\beta}}{(1+s^4)^{1/\beta}}ds<\infty,$$

and from (5.2) *and* (5.3) *we conclude that $(H3)$ is verified for the kernel $a(t)$.*

$(H4)$ *It is clear that f is continuous. Taking $h(t):=1,\ t\geq-\sigma$ and $\mathcal{H}(t):=\ln(1+t),\ t\geq 0$, we obtain that*

$$|f(t,u)-f(t,v)| = \frac{t^2}{1+t^4}\ln\Big(\frac{1+|u(-\pi/2)|}{1+|v(-\pi/2)|}\Big) \leq \ln\Big(1+\frac{|u(-\pi/2)|-|v(-\pi/2)|}{1+|v(-\pi/2)|}\Big)$$
$$\leq \ln(1+|u(-\pi/2)-v(-\pi/2)|) \leq \ln(1+\|u-v\|_{C([-\frac{\pi}{2},0],\mathbb{R})})$$
$$= h(t)\mathcal{H}(\|u-v\|_{C([-\frac{\pi}{2},0],\mathbb{R})}).$$

Finally, note by [4, Remark 3.2] *that $\mathcal{H}$ is superadditive.*

(H5) Note that the condition $\delta > 1$ *implies* $A < 1$. *Also*

$$B = \sup_{t \geq 0} \int_0^t |a(t-s)| f(s,0)\, ds = 0.$$

From [4, Lemma 2.1] *we obtain*

$$\lim_{n \to \infty} A^n \mathcal{H}^n(t) = 0.$$

(H6) Since $\frac{\pi}{2}\sqrt{e} = \sup_{-\frac{\pi}{2} \leq t \leq 0} |te^{\frac{t}{\pi}}|$ *we have that for any* $r > 0$ *such that*

$$r \geq \frac{\frac{\pi}{2}\sqrt{e}}{1-A}$$

we get

$$\sup_{-\frac{\pi}{2} \leq t \leq 0} |\phi(t)| + A\mathcal{H}(r) + B \leq \frac{\pi}{2}\sqrt{e} + Ar \leq r.$$

Hence the hypothesis of Theorem 4.2 are fulfilled and consequently the Equation (5.4) *has at least one uniformly locally attractive solution in* $BC([-\frac{\pi}{2}, \infty), \mathbb{R})$ *for any* $\delta > 1$ *and* $\alpha > 0$.

References

[1] M. I. Abbas. On the existence of locally attractive solutions of a nonlinear quadratic Volterra integral equation of fractional order. *Adv. Difference Equ.* 2010, Art. ID 127093, 11 pp.

[2] S. Abbas, M. Benchohra and J.J. Nieto. Global attractivity of solutions for nonlinear fractional order Riemann-Liouville Volterra-Stieltjes partial integral equations. *Electron. J. Qual. Theory Differ. Equ.* **2012** No 81, 15 pp.

[3] R. Agarwal, J. Banaś, Dhage, B.C. and S. Sarkate. Attractivity results for a nonlinear functional integral equation. *Georgian Math.* **18** (2011), 1–19.

[4] A. Aghajani, J. Banaś and N. Sabzali. Some generalizations of Darbo fixed point theorem and applications. *Bull. Belg. Math. Soc. Simon Stevin* **20** (2) (2013), 345–358.

[5] A. Aghajani and Y. Jalilian. Existence and global attractivity of solutions of a nonlinear functional integral equation. *Commun. Nonlinear Sci. Numer. Simul.* **15** (11) (2010), 3306–3312.

[6] A. Aghajani and E. Pourhadi. Application of measure of noncompactness to l_1-solvavility of infinite system of second order differential equations. *Bull. Belg. Math. Soc. Simon Stevin* **22** (1) (2015), 105–118.

[7] A. Aghajani, R. Allahyari and M. Mursaleen. A generalization of Darbo's theorem with application to the solvability of systems of integral equations, *Jour. Comput. Appl. Math.,* **260** (2014) 68–77.

[8] A. Aghajani, M. Mursaleen and A. Shole Haghighi, Fixed point theorems for Meir-Keeler condensing operators via measure of noncompactness, *Acta Math. Sci.,* **35B** (3) (2015) 552–566.

[9] D. Baleanu, S.Z. Nazemi and S. Rezapour. Attractivity for a k-dimensional system of fractional functional differential equations and global attractivity for a k-dimensional system of nonlinear fractional differential equations. *J. Inequal. Appl.* **31,** (2014) 1–14.

[10] K. Balachandran, J.K. Park and M. Diana. On local attractivity of solutions of a functional integral equation of fractional order with deviating arguments. *Comm. Nonlinear Sci. Numer. Simulat.* **15** (2010), 2809–2817.

[11] J. Banaś and M. Mursaleen, Sequence Spaces and Measures of Noncompactness with Applications to Differential and Integral Equations, Springer, New Delhi, 2014.

[12] J. Banaś and K. Goebel. Measure of noncompactness in Banach spaces. Lectures Notes in Pure and Applied Math (Vol 60), Dekker, New York-Basel, 1980.

[13] J. Banaś and D. O'Regan. On existence and local attractivity of solutions of a quadratic Volterra integral equation of fractional order. *J. Math. Anal. Appl.* **345** (2008), 573–582.

[14] J. Banaś and B. Rzepka. An application of measure of noncompactness in the study of asymptotic stability. *Appl. Math. Lett.* **16** (2003), 1–6.

[15] J. Banaś and B. Rzepka. On existence and asymptotic stability of solutions of a nonlinear integral equation. *J. Math. Anal. Appl.* **284** (2003), 165–173.

[16] M. V. Bartuccelli, J.H.B. Deane and G. Gentile. Globally and locally attractive solutions for quasi-periodically forced systems. *J. Math. Anal. Appl.* **328** (1) (2007), 699-714.

[17] F. Chen, J.J. Nieto and Y. Zhou. Global attractivity for nonlinear fractional differential equations. *Nonlinear Anal. Real World Appl.* **13** (2012), 287–298.

[18] F. Chen and Y. Zhou. Attractivity of fractional functional differential equations. *Comp. Math. Appl.* **62**(3) (2011), 1359–1369.
[19] M. Benchohra, J. Henderson, S.K Ntouyas and A. Ouahab. Existence results for fractional order functional differential equations with infinite delay. *J. Math. Anal. Appl.* **338** (2008), 1340–1350.
[20] D. Franco and D. O'Regan. Solutions of Volterra integral equations with infinite delay. *Math. Nachr.* **281** (3) (2008), 325–336.
[21] I.S. Gradshteyn and I.M. Ryzhik. Table of Integrals, Series, and Products. Seventh edition. Elsevier Academic Press, 2007.
[22] G. Gripenberg, S.-O. Londen and O. Staffans. Volterra Integral and Functional Equations. Encyclopedia of Mathematics and its Applications, 34. Cambridge University Press, Cambridge, 1990.
[23] X. Hu and J. Yan. The global attractivity and asymptotic stability of solution of a nonlinear integral equation. *J. Math. Anal. Appl.* **321** (2006), 147–156.
[24] J. Losada, J.J. Nieto and E. Pourhadi. On the attractivity of solutions for a class of multi-term fractional functional differential equations. *J. Comp. Appl. Math.* In press.
[25] F. Mainardi. Fractional Calculus and Waves in Linear Viscoleasticity. An introduction to mathematical models. Imperial College Press, London, 2010.
[26] L. T. P. Ngoc and N. T. Long. On a fixed point theorem of Krasnoselskii type and application to integral equations. *Fixed Point Theory and App.* **2006** (ID 30847), 1–24.
[27] L. Olszowy. Fixed point theorem in the Fréchet space $C(\mathbb{R}_+)$ and functional integral equations on an unbounded interval. *Comp. Math. Appl.* **218** (2012), 9066–9074.
[28] J. Prüss. Evolutionary Integral Equations and Applications. [2012] reprint of the 1993 edition. Modern Birkhäuser Classics. Birkhäuser/Springer Basel AG, Basel, 1993.

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