MILD SOLUTIONS FOR MULTI-TERM TIME-FRACTIONAL DIFFERENTIAL EQUATIONS WITH NONLOCAL INITIAL CONDITIONS

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Abstract. We prove the existence of mild solutions for the multi-term time-fractional order abstract differential equation

$$D_t^{\alpha+1}u(t) + c_1 D_t^{\beta_1}u(t) + ... + c_d D_t^{\beta_k}u(t) = Au(t) + D_t^{\alpha-1}f(t, u(t)), \quad t \in [0,1],$$

with nonlocal initial conditions, where A is the generator of a strongly continuous cosine function, $0 < \alpha \leq \beta_d \leq ... \leq \beta_1 \leq 1$ and $c_k \geq 0$ for all $k = 1, ..., d$.

1. Introduction

Let X be a Banach space. Our concern in this paper is the study of existence of mild solutions for fractional order differential equations of the form

$$D_t^{\gamma}u(t) + \sum_{k=1}^{d} c_k D_t^{\beta_k}u(t) = Au(t) + F(s, u(s)), \quad t \in [0,1], \quad 0 < \gamma \leq 2, \tag{1.1}$$

with prescribed nonlocal initial conditions $u(0) = 0$ and $u'(0) = g(u)$, where $A : D(A) \subset X \to X$ is a closed linear operator, F and g are vector-valued functions, D_t^{γ} denotes the Caputo fractional derivative of order γ, and β_k are positive real numbers.

Fractional order differential equations represent a subject of interest in different context and areas of research, see e.g. [1, 3, 5, 7, 8, 11, 16, 17], the survey paper [6], and the references therein.

Multi-term time-fractional differential equations increasingly begin to receive attention of a number of authors. For instance, in the papers [13] and [10] a two-term time fractional differential equation, which includes a concrete case of fractional diffusion-wave problem, is studied in the abstract context. On the other hand, the case of the multi-term time-fractional diffusion-wave equation with the constant coefficients was recently considered in [4]. In the paper [15], a general class of multi-term time-fractional diffusion equations with variable coefficients is considered. In particular, the notion of the generalized solution of the initial-boundary-value problem for the generalized multi-term time-fractional diffusion equation is introduced and some existence results for the generalized solution are given. In the paper [9], analytical solutions for a multi-term time-fractional diffusion-wave equation was analyzed and in the paper [12], the authors present numerical methods for

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the solution of time-fractional diffusion equations where the fractional differential operator with respect to the time variable is assumed to be of Caputo type and to have a multi-term structure.

The equation (1.1) is a general model that include recent investigations in the subject. Indeed, in the interesting paper [14] the authors C. Li, M. Kostic, M. Li and S. Piskarev studied the equation (1.1) with $\gamma = \alpha$, $\alpha > \beta_1 > \cdots > \beta_d$, and initial conditions. They have obtained existence of resolvent families, algebraic equations, approximations and a complex inversion formulae by means of constructive arguments based on Laplace transform theory. On other hand, in the reference [13] the author studied mild solutions for the equation (1.1) with $\gamma = \alpha + 1$, $c_1 = \mu$, $c_2 = \cdots = c_d = 0$ and nonlocal conditions. Then, it is natural to ask: Under which conditions mild solutions for the general equation (1.1) with nonlocal initial conditions exists ?. In this paper, we answer such question finding a subordination condition on the indexes of the time-fractional derivatives, and assuming that the operator A is the generator of a *bounded* cosine operator function. It is remarkable that our condition contrasts with those hypothesis used in [13] where it is assumed that A is sectorial, i.e. the generator of an analytic semigroup. From a certain perspective, our condition seems to be more natural in the sense that equation (1.1) represents fractional oscillation for $1 < \gamma \leq 2$. See Theorem 3.5 below. As in [13], we use a method based on operator theory, which consist in the construction of a family of strongly continuous operators whose properties are analogous to the theory of C_0-semigroups. Indeed, it corresponds to an extension of such theory and has been proposed in the recent reference [14].

The outline of this paper is as follows: In the second section, we fix some notation and basic notions on fractional derivatives and Laplace transforms. The third section, deals with a notion - introduced in [14] - of family of bounded and linear operators defined on a Banach space X which provides the right framework for the analysis of the given abstract fractional differential equation by means of an operator-theoretical approach, in the same spirit of the well known theory of C_0-semigroups and their correspondence with the abstract Cauchy problem of first order. The novelty here is our assumption on the operator A, because we assume that such operator is the generator of a bounded strongly continuous cosine function, which is a typical choice in hyperbolic problems. Moreover, we prove in this section that this class of operators A (generators of cosine functions) are contained in the more general class of operators defined in section 3 (see Theorem 3.4 below). Finally, the last section 4, deals with the main result of this paper, concerning existence of mild solutions for the semilinear given problem. Here the main novelty is that no additional hypothesis on the qualitative behaviour of the family of operators generated by A is needed, such as e.g. compactness, because more regularity is automatically obtained thanks to the representation of the mild solution by means of a kind of variation of parameters formula (see formula (3.6) below). Finally, our main theorem in this section is Theorem 3.5, which extends to the general case presented here, the main result in the article [13]. We finish this paper with an illustrative example.

2. Preliminaries

Let $\alpha > 0$ be given. We define

$$g_\alpha(t) := \begin{cases} \dfrac{1}{\Gamma(\alpha)} t^{\alpha-1}, & t > 0 \\ 0, & t \leq 0, \end{cases}$$

where Γ is the Gamma function. These functions satisfy the following properties $g_\alpha * g_\beta = g_{\alpha+\beta}$, for $\alpha, \beta > 0$ and $\widehat{g_\alpha}(\lambda) = \frac{1}{\lambda^\alpha}$ for $Re\lambda > 0$ and $\alpha > 0$. Here, the hat $\hat{\cdot}$ denotes Laplace transform. Recall that for a locally integrable and exponentially bounded function $f : \mathbb{R}_+ \to X$ (i.e. there exists $M > 0$ and $\omega \in \mathbb{R}$ such that $\|f(t)\| \leq Me^{\omega t}$) the Laplace transform

$$\hat{f}(\lambda) := \int_0^\infty e^{-\lambda s} f(s) ds,$$

exists for $Re(\lambda) > \omega$. We also recall the following definitions.

Definition 2.1. *Let $f : \mathbb{R}_+ \to X$ be a locally integrable function and $\alpha > 0$. The Riemann-Liouville fractional integral of order $\alpha > 0$ is defined as follows:*

$$J_t^\alpha f(t) := (g_\alpha * f)(t) = \int_0^t g_\alpha(t-\tau) f(\tau)\, d\tau, \quad t > 0,\ \alpha > 0; \tag{2.1}$$

and $J_0^\alpha f(t) := f(t)$.

This integral satisfy the following properties $J_t^\alpha \circ J_t^\beta = J_t^{\alpha+\beta}$ and $\widehat{J_t^\alpha f}(\lambda) = \frac{1}{\lambda^\alpha} \widehat{f}(\lambda)$ for $Re(\lambda) > 0$. We denote

$$D_t^n f(t) := \frac{d^n}{dt^n} f(t), \text{ for } n \in \mathbb{N}.$$

Then

$$(D_t^n \circ J_t^n) f(t) = f(t), \quad t > 0;$$

and

$$(J_t^n \circ D_t^n) f(t) = f(t) - \sum_{k=0}^{n-1} \frac{f^{(k)}(0)}{k!} t^k, \quad t > 0,\ n \in \mathbb{N}.$$

In particular, if $f(0) = f'(0) = \cdots = f^{(n-1)}(0) = 0$, then

$$(J_t^n \circ D_t^n) f(t) = f(t), \quad t > 0.$$

Definition 2.2. *Let $\alpha > 0$ be given and denote $m = \lceil \alpha \rceil$. The Riemann-Liouville fractional derivative of order $\alpha > 0$ is defined for all $f : \mathbb{R}_+ \to X$ as follows*

$$\mathbb{D}_t^\alpha f(t) := D_t^m (g_{m-\alpha} * f)(t) = D_t^m J_t^{m-\alpha} f(t), \quad m - 1 < \alpha \leq m. \tag{2.2}$$

Furthermore, $\mathbb{D}_t^0 f(t) := f(t)$.

We have the following property

$$(\mathbb{D}_t^\alpha \circ J_t^\alpha) f(t) = f(t), \quad t > 0.$$

Example 2.3. *Let $\alpha \geq 0$ and $\gamma > -1$. Then*

(i) $J_t^\alpha t^\gamma = \dfrac{\Gamma(\gamma+1)}{\Gamma(\gamma+1+\alpha)} t^{\gamma+\alpha}, \quad t > 0;$

(ii) $J_t^\alpha g_\gamma(t) = g_{\gamma+\alpha}(t), \quad t > 0;$

(iii) $\mathbb{D}_t^\alpha t^\gamma = \dfrac{\Gamma(\gamma+1)}{\Gamma(\gamma+1+\alpha)} t^{\gamma-\alpha}, \quad t > 0.$

Definition 2.4. *Let $\alpha > 0$ be given and denote $m = \lceil \alpha \rceil$. The Caputo fractional derivative of order $\alpha > 0$ is defined by*

$$D_t^\alpha f(t) := J_t^{m-\alpha} D_t^m f(t) = (g_{m-\alpha} * D_t^m) f(t) = \int_0^t g_{m-\alpha}(t-\tau) \frac{d^m}{dt^m} f(\tau)\, d\tau. \tag{2.3}$$

Note that $f(0) = f'(0) = \cdots = f^{(m-1)}(0) = 0$ is a necessary condition for the equality between the Riemann-Liouville and Caputo derivative, that is

$$\mathbb{D}_t^\alpha f(t) = D_t^\alpha f(t), \quad t > 0.$$

Finally, we recall the following property concerning the Laplace transform. Let $m - 1 < \alpha \le m$. Then

$$(J_t^\alpha \circ D_t^\alpha) f(t) = f(t) - \sum_{k=0}^{m-1} f^{(k)}(0) g_{k+1}(t). \tag{2.4}$$

and

$$\widehat{D_t^\alpha f}(\lambda) = \lambda^\alpha \widehat{f}(\lambda) - \sum_{k=0}^{m-1} f^{(k)}(0) \lambda^{\alpha-1-k}. \tag{2.5}$$

Remark 2.5. *If $f(0) = f'(0) = ... = f^{(m-1)}(0) = 0$, then $J_t^\alpha D_t^\alpha f(t) = f(t)$ and $\widehat{D_t^\alpha f}(\lambda) = \lambda^\alpha \hat{f}(\lambda)$.*

3. Mild solutions and families of linear operators

We consider the linear equation

$$D_t^{\alpha+1} u(t) + \sum_{k=1}^{d} c_k D_t^{\beta_k} u(t) = Au(t) + h(t), \quad t \ge 0. \tag{3.1}$$

Our objective in this section is give a representation of the solution in terms of certain family of bounded and linear operators defined below. The obtained representation will be then used to give an appropriate definition of mild solution for the associated semilinear problem.

Definition 3.1. [14] *Let $\alpha > 0, \beta_k, c_k$ be real numbers and let A be a closed linear operator with domain $D(A)$ on a Banach space X. We call A the generator of an (α, β_k)-resolvent family if there exist $\omega \ge 0$ and a strongly continuous function $S_{\alpha,\beta_k} : \mathbb{R}^+ \to \mathcal{B}(X)$ such that $\{\lambda^{\alpha+1} + \sum_{k=1}^d c_k \lambda^{\beta_k} : Re\lambda > \omega\} \subset \rho(A)$ and*

$$\lambda^\alpha \left(\lambda^{\alpha+1} + \sum_{k=1}^{d} c_k \lambda^{\beta_k} - A \right)^{-1} x = \int_0^\infty e^{-\lambda t} S_{\alpha,\beta_k}(t) x\, dt, \quad Re\lambda > \omega, \quad x \in X. \tag{3.2}$$

Now we consider the initial valued problem

$$
\begin{cases} D_t^{\alpha+1}u(t) + c_1 D_t^{\beta_1}u(t) + c_2 D_t^{\beta_2}u(t) + \cdots + c_d D_t^{\beta_d}u(t) = Au(t) + h(t), \quad t \in [0,1], \\ u(0) = x_0, \\ u'(0) = x_1 \end{cases} \tag{3.3}
$$

where $0 < \alpha \le \beta_d \le ... \le \beta_1 \le 1$.

By taking Riemann-Liouville integral of order $\alpha + 1$ in the Equation (3.3) we have

$$
\begin{aligned} J_t^{\alpha+1}D_t^{\alpha+1}u(t) + c_1 J_t^{\alpha+1}D_t^{\beta_1}u(t) + c_2 J_t^{\alpha+1}D_t^{\beta_2}u(t) \\ + \cdots + c_d J_t^{\alpha+1}D_t^{\beta_d}u(t) = J_t^{\alpha+1}Au(t) + J_t^{\alpha+1}h(t). \end{aligned}
$$

Since $\alpha + 1 - \beta_k > 0$ and $\beta_k > 0$ for all $k = 1, ..., d$, then $J_t^{\alpha+1} = J_t^{\alpha+1-\beta_k}J_t^{\beta_k}$ for all $k = 1, 2, ..., d$. Hence we can rewrite the preceding equation as

$$
\begin{aligned} J_t^{\alpha+1}D_t^{\alpha+1}u(t) + c_1 J_t^{\alpha+1-\beta_1}(J_t^{\beta_1}D_t^{\beta_1}u(t)) + c_2 J_t^{\alpha+1-\beta_2}(J_t^{\beta_2}D_t^{\beta_2}u(t)) \\ + \cdots + c_d J_t^{\alpha+1-\beta_d}(J_t^{\beta_d}D_t^{\beta_d}u(t)) = J_t^{\alpha+1}Au(t) + J_t^{\alpha+1}h(t). \end{aligned}
$$

Now, applying the definition of the Riemann-Liouville integral and the identity (2.4) we obtain

$$
\begin{aligned} u(t) - \sum_{j=0}^{\lceil \alpha+1 \rceil - 1} g_{j+1}(t)u^{(j)}(0) + \sum_{k=1}^{d} c_k J_t^{\alpha+1-\beta_k}\left(u(t) - \sum_{j=0}^{\lceil \beta_k \rceil - 1} g_{j+1}(t)u^{(j)}(0) \right) \\ = (g_{\alpha+1} * Au)(t) + (g_{\alpha+1} * h)(t). \end{aligned}
$$

Since $\alpha + 1 \le 2$, $\beta_k \le 1$ and $u(0) = x_0$, $u'(0) = x_1$ it follows that $\lceil \alpha + 1 \rceil = 2$ and $\lceil \beta_k \rceil = 1$. Therefore, using (ii) in Example 2.3 we obtain that the equation (2.5) is equivalent to the following integral equation

$$
\begin{aligned} u(t) = g_1(t)x_0 + g_2(t)x_1 - \sum_{k=1}^{d} c_k (g_{\alpha+1-\beta_k} * u)(t) \\ + \sum_{k=1}^{d} c_k g_{\alpha+2-\beta_k}(t)x_0 + A(g_{\alpha+1} * u)(t) + (g_{\alpha+1} * h)(t). \end{aligned} \tag{3.4}
$$

The next theorem guarantees the existence of (α, β_k)-resolvent families.

Theorem 3.2. *Let $0 < \alpha \le \beta_d \le ... \le \beta_1 \le 1$ and $c_k \ge 0$ be given and A be a generator of a bounded and strongly continuous cosine family $\{C(t)\}_{t \in \mathbb{R}}$. Then A generates a bounded (α, β_k)-resolvent family $\{S_{\alpha,\beta_k}(t)\}_{t \ge 0}$.*

Proof. By the subordination principle (see [3, Theorem 3.1]) we have that A generates an $(\alpha + 1)$-times resolvent family given by

$$
S_{\alpha+1}(t)x = \int_0^\infty \frac{1}{t^{(\alpha+1)/2}} \Phi_{(\alpha+1)/2}(ut^{-(\alpha+1)/2})C(u)x\,du, \quad x \in X, \ t > 0,
$$

where

$$\Phi_{\alpha+1}(z) := \sum_{n=0}^{\infty} \frac{(-z)^n}{n!\Gamma(-(\alpha(n+1))-n)}, z \in \mathbb{C},$$

is the Wright's function. From [3, Theorem 3.3]), the family $S_{\alpha+1}(t)$ admits analytic extension to the sector $\sum_{(\frac{1-\alpha}{1+\alpha})\frac{\pi}{2}} := \{\lambda \in \mathbb{C} \setminus \{0\} \ : \ |arg(\lambda)| < \frac{\pi}{2}\frac{1-\alpha}{1+\alpha}\}$. The conclusion follows from [14, Theorem 3.7]. For the boundedness, we note that

$$\begin{aligned} \|S_{\alpha+1}(t)x\| &= \int_0^{\infty} \frac{1}{t^{(\alpha+1)/2}} \Phi_{(\alpha+1)/2}(ut^{-(\alpha+1)/2}) \|C(u)x\| du \\ &\le M \int_0^{\infty} \frac{1}{t^{(\alpha+1)/2}} \Phi_{(\alpha+1)/2}(ut^{-(\alpha+1)/2}) du \|x\| \\ &= M \int_0^{\infty} \Phi_{(\alpha+1)/2}(s) ds \|x\| \le C\|x\|, \end{aligned}$$

for all $x \in X$, proving the theorem. □

With the goal to construct a representation of the solution for the problem (3.3) in terms of the family $\{S_{\alpha,\beta_k}(t)\}_{t\ge 0}$, we apply the Laplace transform method. Then we obtain

$$\lambda^{\alpha+1}\widehat{u}(\lambda) - \sum_{j=0}^{\lceil \alpha+1 \rceil -1} u^{(j)}(0)\lambda^{\alpha-j} + \sum_{k=1}^{d} c_k \left[\lambda^{\beta_k}\widehat{u}(\lambda) - \sum_{j=0}^{\lceil \beta_k \rceil -1} u^{(j)}(0)\lambda^{\beta_k-1-j} \right] = A\widehat{u}(\lambda) + \widehat{h}(\lambda).$$

Applying the given initial conditions, we have

$$\lambda^{\alpha+1}\widehat{u}(\lambda) - \lambda^{\alpha}x_0 - \lambda^{\alpha-1}x_1 + \sum_{k=1}^{d} c_k\lambda^{\beta_k}\widehat{u}(\lambda) - \sum_{k=1}^{d} c_k\lambda^{\beta_k-1}x_0 = A\widehat{u}(\lambda) + \widehat{h}(\lambda).$$

This is equivalent to

$$\left(\lambda^{\alpha+1} + \sum_{k=1}^{d} c_k\lambda^{\beta_k} - A\right)\widehat{u}(\lambda) = \lambda^{\alpha}x_0 + \lambda^{\alpha-1}x_1 + \sum_{k=1}^{d} c_k\lambda^{\beta_k-1}x_0 + \widehat{h}(\lambda).$$

Hence, assuming existence of the family $S_{\alpha,\beta_k}(t)$ we obtain

$$\begin{aligned} \widehat{u}(\lambda) = {} & \lambda^{\alpha}\left(\lambda^{\alpha+1} + \sum_{k=1}^{d} c_k\lambda^{\beta_k} - A\right)^{-1} x_0 + \lambda^{\alpha-1}\left(\lambda^{\alpha+1} + \sum_{k=1}^{d} c_k\lambda^{\beta_k} - A\right)^{-1} x_1 \\ & + \sum_{k=1}^{d} c_k\lambda^{\beta_k-1}\left(\lambda^{\alpha+1} + \sum_{k=1}^{d} c_k\lambda^{\beta_k} - A\right)^{-1} x_0 + \left(\lambda^{\alpha+1} + \sum_{k=1}^{d} c_k\lambda^{\beta_k} - A\right)^{-1} \widehat{h}(\lambda). \end{aligned}$$

Equivalently,

$$u(t) = S_{\alpha,\beta_k}(t)x_0 + (1 * S_{\alpha,\beta_k})(t)x_1 + \sum_{k=1}^{d} c_k(g_{\alpha+1-\beta_k} * S_{\alpha,\beta_k})(t)x_0 + (g_\alpha * S_{\alpha,\beta_k} * h)(t). \tag{3.5}$$

In particular, for $x_0 = 0$ and $x_1 = g(u)$ we have

$$u(t) = (1 * S_{\alpha,\beta_k})(t)g(u) + (g_\alpha * S_{\alpha,\beta_k} * h)(t), \quad t > 0. \tag{3.6}$$

The above representation formula allows to give the following definition.

Definition 3.3. *We say that a function $u : \mathbb{R}_+ \to X$ is a mild solution of the equation*

$$D_t^{\alpha+1}u(t) + c_1 D_t^{\beta_1}u(t) + c_2 D_t^{\beta_2}u(t) + \cdots c_d D_t^{\beta_d}u(t) = Au(t) + D_t^{\alpha-1}f(t, u(t)), \tag{3.7}$$

with nonlocal initial conditions $u(0) = 0, u'(0) = g(u)$ if it satisfies the formula

$$u(t) = (1 * S_{\alpha,\beta_k})(t)g(u) + \int_0^t (1 * S_{\alpha,\beta_k})(t-s)f(s, u(s))ds, \quad t > 0. \tag{3.8}$$

We next use the Hausdorff measure of noncompactness and a fixed point argument to prove the existence of a mild solution for the equation (3.7) where $f : I \times X \to X$ and $g : C([0,1]; X) \to X$ are suitable functions.

Remark 3.4. *Let $S_{\alpha,\beta_k}(t)$ be the family generated by the operator A in the Theorem 3.2. Since $S_{\alpha,\beta_k}(t)$ is bounded, then the function $t \to g_1 * S_{\alpha,\beta_k}(t)$ is norm continuous for $t > 0$. Indeed, we have for $0 < t < s$ that*

$$\left\| \int_0^t S_{\alpha,\beta_k}(\tau)\, d\tau - \int_0^s S_{\alpha,\beta_k}(\tau) \right\| \leq \int_t^s \|S_{\alpha,\beta_k}(\tau)\|\, d\tau \leq \sup_{\tau \geq 0} \|S_{\alpha,\beta_k}(\tau)\| |t - s|$$

We will denote $M := \sup\{||g_1 * S_{\alpha,\beta_k}(t)|| : t \in [0,1]\}$.

In order to give the main result of this section, we consider the following assertions.

(H1) A is the generator of a bounded strongly continuous cosine family.

(H2) $g : C([0,1]; X) \to X$ is continuous, compact and there exists positive constants c and d such that $\|g(u)\| \leqslant c\|u\| + d$, $\forall u \in C([0,1]; X)$.

(H3) $f : [0,1] \times X \to X$ satisfies the Carathéodory type conditions, that is, $f(\cdot, x)$ is measurable for all $x \in X$ and $f(t, \cdot)$ is continuous for almost all $t \in [0,1]$.

(H4) There exists a function $m \in L^1(0,1;\mathbb{R}^+)$ (here $L^1(0,1;\mathbb{R}^+)$ is the space of $\mathbb{R}^+$-valued Bochner functions on $[0,1]$ with the norm $\|x\| = \int_0^1 \|x(s)\|ds$) and a nondecreasing continuous function $\Phi : \mathbb{R}^+ \to \mathbb{R}^+$ such that

$$\|f(t,x)\| \leqslant m(t)\Phi(\|x\|)$$

for all $x \in X$ and almost all $t \in [0,1]$.

(H5) There exists a function $H \in L^1(0,1;\mathbb{R}^+)$ such that for any bounded $B \subseteq X$

$$\gamma(f(t,B)) \leqslant H(t)\gamma(B)$$

for almost all $t \in [0,1]$.

In $(H5)$ γ denote the Hausdorff measure of noncompactness which is defined by

$$\gamma(B) = \inf\{\epsilon > 0 : B \text{ has a finite cover by balls of radius } \epsilon\}.$$

We note that this measure of noncompactness satisfies interesting regularity properties. For more information, we refer to [2]. We are now in position to establish our main result.

Theorem 3.5. *Let* $0 < \alpha \leq \beta_d \leq ... \leq \beta_1 \leq 1$ *and* $c_k \geq 0$ *be given. If the hypothesis (H1)-(H5) are satisfied and there exists a constant* $R > 0$ *such that*

$$M(cR + d) + M\Phi(R) \int_0^1 m(s)ds \leqslant R$$

then the problem (3.7) *has at least one mild solution.*

Proof. Define $F : C([0,1]; X) \to C([0,1]; X)$ by

$$(Fx)(t) = (1 * S_{\alpha,\beta_k})(t)g(x) + \int_0^t (1 * S_{\alpha,\beta_k})(t-s)f(s, x(s))ds, \quad t \in [0,1].$$

First, we show that F is a continuous map. Let $\{x_n\}_{n\in\mathbb{N}} \subseteq C([0,1]; X)$ be a sequence such that $x_n \to x$ (in the norm of $C([0,1]; X)$). Note that

$$\|F(x_n) - F(x)\| \leqslant M\|g(x_n) - g(x)\| + M \int_0^1 \|f(s, x_n(s)) - f(s, x(s))\| ds. \tag{3.9}$$

By the dominated convergence Theorem and assumptions (H1) and (H2) we conclude that $\|F(x_n) - F(x)\| \to 0$ as $n \to \infty$.
Let $B_R := \{x \in C([0,1]; X) : \|x(t)\| \leqslant R$ for all $t \in [0,1]\}$. Is clear that B_R is bounded and convex.

For any $x \in B_R$ we have

$$\begin{aligned}\|(Fx)(t)\| &\leqslant \|S_{\alpha,\beta_k}(t)g(x)\| + \left\| \int_0^t S_{\alpha,\beta_k}(t-s)f(s, x(s))ds \right\| \\ &\leqslant M(cR + d) + M\Phi(R) \int_0^1 m(s)ds \leqslant R.\end{aligned}$$

Therefore $F : B_R \to B_R$ is a bounded operator and $F(B_R)$ is a bounded set. Moreover, by norm continuity of the function $t \to (1 * S_{\alpha,\beta_k})(t)$ we have that $F(B_R)$ is an equicontinuous set of functions. Define $B := \overline{co}(F(B_R))$. Then B is an equicontinuous set of functions and $F : B \to B$ is a continuous operator.

Let $\varepsilon > 0$. By [18, Lemma 2.4] there exists $\{y_n\}_{n\in\mathbb{N}} \subset F(B)$ such that

$$\begin{aligned}
\gamma(FB(t)) &\leqslant 2\gamma(\{y_n(t)\}_{n\in\mathbb{N}}) + \varepsilon \\
&\leqslant 2\gamma\left(\int_0^t S_{\alpha,\beta}(t-s)f(s,\{y_n(s)\}_{n\in\mathbb{N}})ds\right) + \varepsilon \\
&\leqslant 4M\int_0^t \gamma(f(s,\{y_n(s)\})\}_{n\in\mathbb{N}})ds + \varepsilon \\
&\leqslant 4M\int_0^t H(s)\gamma(\{y_n(s)\}_{n\in\mathbb{N}})ds + \varepsilon \\
&\leqslant 4M\gamma(\{y_n\})\int_0^t H(s)ds + \varepsilon \\
&\leqslant 4M\gamma(B)\int_0^t H(s)ds + \varepsilon.
\end{aligned} \tag{3.10}$$

Since $H \in L^1(0,1;X)$ there exists $\varphi \in C([0,1];\mathbb{R}_+)$ such that

$$\int_0^1 |H(s) - \varphi(s)|ds < \alpha, \quad \left(\alpha < \frac{1}{4M}\right).$$

Let $N := \max\{\varphi(t) : t \in [0,1]\}$. Then

$$\begin{aligned}
\gamma(FB(t)) &\leqslant 4M\gamma(B)\left[\int_0^t |H(s) - \varphi(s)|ds + \int_0^t \varphi(s)ds\right] + \varepsilon \\
&\leqslant 4M\gamma(B)\left[\alpha + Nt\right] + \varepsilon.
\end{aligned}$$

Since $\varepsilon > 0$ is arbitrary we obtain that

$$\gamma(FB(t)) \leqslant (a + bt)\gamma(B) \tag{3.11}$$

where $a = 4\alpha M$ and $b = 4MN$.

Let $\varepsilon > 0$, by [18, Lemma 2.4] there exists $\{y_n\}_{n\in\mathbb{N}} \subseteq \overline{co}(F(B))$ such that

$$\begin{aligned}
\gamma(F^2(B(t))) &\leqslant 2\gamma\left(\int_0^t S_{\alpha,\beta_k}(t-s)f(s,\{y_n(s)\}_{n\in\mathbb{N}})ds\right)+\varepsilon\\
&\leqslant 4M\int_0^t \gamma(f(s,\{y_n(s)\}_{n\in\mathbb{N}}))ds+\varepsilon\\
&\leqslant 4M\int_0^t H(s)\gamma(\overline{co}(F^1B(s)))ds+\varepsilon\\
&\leqslant 4M\int_0^t H(s)\gamma(F^1B(s))ds+\varepsilon\\
&\leqslant 4M\int_0^t \big[|H(s)-\varphi(s)|+\varphi(s)\big](a+bs)\gamma(B)ds+\varepsilon\\
&\leqslant 4M(a+bt)\int_0^t |H(s)-\varphi(s)|ds+4MN\left(at+\frac{bt^2}{2}\right)+\varepsilon\\
&\leqslant a(a+bt)+b\left(at+\frac{bt^2}{2}\right)+\varepsilon.
\end{aligned}$$

Since $\varepsilon>0$ is arbitrary then

$$\gamma(F^2(B(t)))\leqslant\left(a^2+2bt+\frac{(bt)^2}{2}\right)\gamma(B). \tag{3.12}$$

By an iterative process we obtain

$$\gamma(F^n(B(t)))\leqslant\left(a^n+C_n^1a^{n-1}bt+C_n^2a^{n-2}\frac{(bt)^2}{2!}+\cdots+\frac{(bt)^n}{n!}\right)\gamma(B). \tag{3.13}$$

By [18, Lemma 2.1] we obtain that

$$\gamma(F^n(B))\leqslant\left(a^n+C_n^1a^{n-1}b+C_n^2a^{n-2}\frac{b^2}{2!}+\cdots+\frac{b^n}{n!}\right)\gamma(B). \tag{3.14}$$

From [18, Lemma 2.5] we know that there exists $n_0\in\mathbb{N}$ such that

$$\left(a^{n_0}+C_{n_0}^1a^{n_0-1}b+C_{n_0}^2a^{n_0-2}\frac{b^2}{2!}+\cdots+\frac{b^{n_0}}{n_0!}\right)=r<1. \tag{3.15}$$

We conclude that

$$\gamma(F^{n_0}B)\leqslant r\gamma(B). \tag{3.16}$$

By [18, Lemma 2.6] , F has a fixed point in B, and this fixed point is a mild solution of equation (3.7). $\square$

4. Example

In this section, we give a simple example to illustrate the feasibility of the assumptions made. Set $X = L^2(\mathbb{R}^d)$, and let $\epsilon > 0$ and $\beta_i > 0$ for every $i = 1, 2, ..., d$ be given, satisfying $0 < \alpha \leq \beta_d \leq ... \leq \beta_1 \leq 1$. We consider the following equation

$$\begin{cases} \partial_t^{\alpha+1}u(t) + c_1\partial_t^{\beta_1}u(t) + c_2\partial_t^{\beta_2}u(t) + \cdots + c_d\partial_t^{\beta_d}u(t) = \Delta u(t) + \partial_t^{\alpha-1}[t^{-\frac{1}{3}}\sin(u(t))], \quad t \in [0,1], \\ u(0,x) = 0, \\ u_t(0,x) = \sum_{i=1}^d \int_{\mathbb{R}^d} \epsilon k(x,y)u(t_i,y)\,dy, \quad x \in \mathbb{R}^d. \end{cases} \tag{4.1}$$

where $0 < t_1 < \cdots < t_d < 1$; $k(x,y) \in L^2(\mathbb{R}^d \times \mathbb{R}^d; \mathbb{R}^+)$, and Δ is the Laplacian with maximal domain $\{v \in X : v \in H^2(\mathbb{R}^d)\}$. Then the equation (4.1) takes the form

$$\begin{cases} D_t^{\alpha+1}u(t) + c_1D_t^{\beta_1}u(t) + c_2D_t^{\beta_2}u(t) + \cdots + c_dD_t^{\beta_d}u(t) = \Delta u(t) + D_t^{\alpha-1}f(t,u(t)), \quad t \in [0,1], \\ u(0) = 0, \\ u'(0) = g_\epsilon(u). \end{cases} \tag{4.2}$$

where the function $g_\epsilon : C([0,1],X) \to X$ is given by $g_\epsilon(u)(x) = \epsilon\sum_{i=1}^m k_g u(t_i)(x)$ with $(k_g v)(x) = \int_{\mathbb{R}^d} k(x,y)v(y)\,dy$, for $v \in X, x \in \mathbb{R}^d$, and the function $f : [0,1] \times X \to X$ is defined by $f(t,u(t)) = t^{-\frac{1}{3}}\sin(u(t))$. Observe that $\|f(t,u(t)) - f(t,v(t))\| \leq t^{-1/3}\|u - v\|$, and hence f satisfies $(H3)$. Note that $\|g_\epsilon(v)\| \leq d\left(\int_{\mathbb{R}^d}\int_{\mathbb{R}^d} \epsilon k^2(z,y)\,dy\,dz\right)^{1/2}\|v\|$, and the function k_g is completely continuous. It proves $(H2)$. In addition $\|f(t,u(t))\| \leq Ct^{-1/3}\Phi(\|u\|)$ with $\Phi(\|u\|) \equiv 1$, proving $(H4)$. Finally, given a bounded subset B of X, and from properties of γ, we obtain that $\gamma(f(t,B)) \leq t^{-1/2}\gamma(\sin(B)) \leq Ct^{-1/2}\gamma(B)$ for some constant $C > 0$ and therefore $(H5)$ is also satisfied.

On the other hand, it follows from theory of cosine families that Δ generates a bounded cosine function $\{C(t)\}_{t\geq 0}$ on $L^2(\mathbb{R}^d)$. By Theorem 3.2, the operator A in equation (4.2) generates a bounded (α, β_k)-times resolvent family $\{S_{\alpha,\beta_k}(t)\}_{t\geq 0}$. Let $K = \sup\{\|g_1 * S_{\alpha,\beta_k}\| : t \in [0,1]\}$. Observe that there exist $\epsilon > 0$ such that $Kc < 1$ where $c = \epsilon d\left(\int_{\mathbb{R}^d}\int_{\mathbb{R}^d} k^2(z,y)\,dy\,dz\right)^{1/2}$. Therefore, there exist $R > 0$ such that $KcR + \frac{3K}{2} < R$. It follows that equation (4.1) has at least a mild solution for all $\epsilon > 0$ sufficiently small.

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