PSEUDO ASYMPTOTIC SOLUTIONS OF FRACTIONAL ORDER SEMILINEAR EQUATIONS

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Abstract. Using a generalization of the semigroup theory of linear operators, we prove existence and uniqueness of mild solutions for the semilinear fractional order differential equation

$$D_t^{\alpha+1}u(t) + \mu D_t^{\beta}u(t) - Au(t) = f(t, u(t)), \quad t > 0, \quad 0 < \alpha \le \beta \le 1,\, \mu \ge 0,$$

with the property that the solution can be written as $u = f + h$ where f belongs to the space of periodic (resp. almost periodic, compact almost automorphic, almost automorphic) functions and h belongs to the space $P_0(\mathbb{R}_+, X) := \{\phi \in BC(\mathbb{R}_+, X) : \lim_{T\to\infty} \frac{1}{T}\int_0^T ||\phi(s)||ds = 0\}$. Moreover, this decomposition is unique.

1. Introduction

Our concern in this paper is the existence, uniqueness and regularity of bounded solutions for fractional order differential equations of the form

$$D_t^{\alpha+1}u(t) + \mu D_t^{\beta}u(t) - Au(t) = f(t, u(t)), \quad t > 0, \tag{1.1}$$

with prescribed initial conditions $u(0)$ and $u'(0)$, and where $A : D(A) \subset X \to X$ is sectorial of angle $\beta\pi/2$, f is a vector-valued function, and D_t^{γ} denotes the Caputo fractional derivative of order γ.

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Fractional order differential equations represent a subject of increasing interest in different contexts and areas of research, see e.g. [1, 3, 13, 14, 16, 27, 29], the survey paper [11] and the references therein. Our motivation to study equation (1.1) comes from recent investigations on the subject. Indeed, in the article [24] the author studied existence and uniqueness of solutions for the abstract equation (1.1) in the special case $\alpha = \beta$ and in the article [31] the authors studied the nonlinear two-term time fractional diffusion wave equation (1.1) with $0 < \alpha < \beta - 1$ and $A = \frac{d^2}{dx^2}$.

In the recent paper [15], asymptotic behavior for mild solutions of (1.1) was studied. However, to the best of our knowledge, no study has investigated the existence and uniqueness of pseudo asymptotic mild solutions for equation (1.1).

The concept of pseudo asymptotic solutions, which is the central subject in this paper, was introduced by Zhang [33], [34], [35] for almost periodic functions in the early nineties. Since then, such a notion became of great interest. For more on the concepts of pseudo-almost periodicity, pseudo-almost automorphy and related issues, we refer the reader to [32] and [17].

In [15] the authors proved that it is possible to give an abstract operator approach to equation (1.1) by defining first an ad-hoc solution family of strongly continuous operators $S_{\alpha,\beta}(t)$ for (1.1) in case $f \equiv 0$. It turns out, that it is a particular case of an (a,k)-regularized family [19] and a generalization of the semigroup theory. Then, the solution of equation (1.1) can be written in terms of a kind of variation of constants formula. It give us the necessary framework to apply an operator theoretical approach in the analysis of pseudo asymptotic solutions for the abstract fractional order differential equation (1.1).

We outline the plan of the paper as follows. In section 2, we recall the concept of fractional order derivatives and some properties of $(\alpha,\beta)_\mu$-regularized families. In section 3 we consider the linear case, that is $f(t,u(t)) = f(t)$ and show existence and uniqueness of pseudo asymptotic solutions of our problem. The existence, uniqueness and the pseudo asymptotic behavior of mild solutions of the semi-linear problem is investigated in Section

4. Existence is proved by means of the contraction mapping theorem. Finally, we conclude the paper by giving a concrete example where the situation in the previous sections can be applied.

2. Preliminaries

Let $\alpha > 0$, $m = \lceil \alpha \rceil$ and $u : [0, \infty) \to X$, where X is a complex Banach space. We denote by $\mathbb{R}_+$ the closed interval $[0, \infty)$. The Caputo fractional derivative of $u \in C(\mathbb{R}_+)$ of order α is defined by

$$D_t^\alpha u(t) := \int_0^t g_{m-\alpha}(t-s)u^{(m)}(s)ds, \quad t > 0,$$

where $g_\beta(t) := \frac{t^{\beta-1}}{\Gamma(\beta)}$, $t > 0$, $\beta > 0$, and in case $\beta = 0$ we set $g_0(t) := \delta_0$, the Dirac measure concentrated at the origin. When $\alpha = n$ is integer, we define $D_t^n := \frac{d^n}{dt^n}, n \in \mathbb{N}$.

We denote by

$$BC(X) := \{f : \mathbb{R} \to X : f \text{ is continuous}, ||f||_\infty := \sup_{t\in\mathbb{R}} ||f(t)|| < \infty\},$$

the Banach space of X-valued bounded and continuous functions on $\mathbb{R}$, with natural norm.

Now we turn our attention to the family of function spaces built on X and which will play a key role in our study.

Let $P_T(X) := \{f \in BC(X) : \ f(t+T) = f(t) \ \forall t \in \mathbb{R}\}$ be the space of all vector-valued periodic functions, with fixed period $T > 0$. We denote by $AP(X)$ the space of almost periodic functions (in the sense of Bohr) which consists of all functions $f \in BC(X)$ such that for each $\epsilon > 0$ there exists a $T > 0$ such that every subinterval of $\mathbb{R}$ of length T contains at least one point τ such that $||f(t+\tau)-f(t)||_\infty \le \epsilon$. This definition is equivalent to the so-called Bochner criterion (cf. [26, Theorem 3.1.8]), namely, $f \in AP(X)$ if and only if for every sequence of reals (s'_n) there exists a subsequence (s_n) such that $(f(\cdot+s_n))$ is uniformly convergent on $\mathbb{R}$.

The space of compact almost automorphic functions will be denoted by $AA_c(X)$. Recall that a continuous bounded function f belongs to $AA_c(X)$ if and only if for all sequence (s'_n) of real numbers, there exists a subsequence $(s_n) \subset (s'_n)$ such that $\lim_{n\to\infty} f(t+s_n) =: \overline{f}(t)$ and $\lim_{n\to\infty} \overline{f}(t-s_n) = f(t)$ uniformly over compact subsets of $\mathbb{R}$.

The space of almost automorphic functions is defined as follows

$$AA(X) := \{f \in BC(X) : \text{ for all } (s'_n), \text{ there exists } (s_n) \subset (s'_n) \text{ such that}$$
$$\lim_{n\to\infty} f(t+s_n) =: \overline{f}(t) \text{ and } \lim_{n\to\infty} \overline{f}(t-s_n) = f(t) \forall t \in \mathbb{R}\},$$

and is endowed with the norm $||\cdot||_\infty$. Almost automorphic functions were introduced by Bochner in connection to some aspects of differential geometry [6, 5, 4, 7]. For more details about this topic we refer to the book [26] where the author gave an important overview about the theory of almost automorphic functions and their applications to differential equations. We note that more general classes of function spaces have been introduced and recently applied to semi-linear differential equations (see [18] and references therein).

We have that $P_T(X), AP(X), AA_c(X)$ and $AA(X)$ are Banach spaces with the norm $||\cdot||_\infty$ and the following inclusions hold:

$$P_T(X) \subset AP(X) \subset AA_c(X) \subset AA(X) \subset BC(X).$$

Now we define the space $P_0(\mathbb{R}_+, X) := \{f \in BC(\mathbb{R}_+, X) : \lim_{T\to\infty} \frac{1}{T}\int_0^T ||f(s)||ds = 0\}$, and define the space of pseudo asymptotically periodic functions as $PP_T(\mathbb{R}_+, X) := P_T(X) \oplus P_0(\mathbb{R}_+, X)$. Analogously, we define the space of pseudo asymptotically almost periodic functions $PAP(\mathbb{R}_+, X) := AP(X) \oplus P_0(\mathbb{R}_+, X)$, the space of pseudo asymptotically compact almost automorphic functions, $PAA_c(\mathbb{R}_+, X) := AA_c(X) \oplus P_0(\mathbb{R}_+, X)$, and the space of pseudo asymptotically almost automorphic functions $PAA(\mathbb{R}_+, X) := AA(X) \oplus P_0(\mathbb{R}_+, X)$.

We have the following natural inclusions

$$PP_T(\mathbb{R}_+, X) \subset PAP(\mathbb{R}_+, X) \subset PAA_c(\mathbb{R}_+, X) \subset PAA(\mathbb{R}_+, X) \subset BC(\mathbb{R}_+, X).$$

Note that all the inclusions are proper. Let $\Lambda \in \{P_T(X), AP(X), AA_c(X), AA(X)\}$.

Definition 2.1. *We say that a function u is a pseudo asymptotic solution of the equation* (1.1) *if u is a solution and belongs to any of the spaces $PP_T(\mathbb{R}_+, X)$, $PAP(\mathbb{R}_+, X)$, $PAA_c(\mathbb{R}_+, X)$ or $PAA(\mathbb{R}_+, X)$.*

Lemma 2.2. *Let X be a Banach space, $h \in L^1_{loc}(\mathbb{R}_+, X)$. If $\lim_{t\to\infty} \|h(t)\| = 0$ then $h \in P_0(\mathbb{R}_+, X)$.*

Proof. We apply the Theorem 4.1.2 of [2] to the function $f(t) := \|h(t)\|$ and obtain the conclusion of the lemma. □

Lemma 2.3. *Let $\{S(t)\}_{t\geq 0} \subset \mathcal{L}(X)$ be a uniformly integrable and strongly continuous family. Let $g \in \Lambda$ and set $z(t) := \int_{-\infty}^0 S(t-s)g(s)\,ds$. Then $z \in P_0(\mathbb{R}_+, X)$.*

Proof.

$$\|z(t)\| = \left\| \int_{-\infty}^0 S(t-s)g(s)\,ds \right\| \leq \int_{-\infty}^0 \|S(t-s)\|\|g(s)\|\,ds \leq \|g\|_\infty \int_t^\infty \|S(s)\|\,ds \to 0, \ (t \to \infty).$$

It follows from Lemma 2.2 that $z \in P_0(\mathbb{R}_+, X)$. □

Lemma 2.4. *Let $\{S(t)\}_{t\geq 0} \subset \mathcal{L}(X)$ be a uniformly integrable and strongly continuous family. If $h \in P_0(\mathbb{R}_+, X)$ then $S * h \in P_0(\mathbb{R}_+, X)$.*

Proof. Let $h \in P_0(\mathbb{R}_+, X)$. Note that the function defined by $\varphi_T(s) := \frac{1}{T}\int_0^{T-s} \|h(u)\|\,du$ is decreasing on $\mathbb{R}_+$. Furthermore, $\varphi_T(0) = \frac{1}{T}\int_0^T \|h(u)\|\,du \to 0$ as $T \to \infty$ by hypothesis.

From Fubini's Theorem we have that

$$\begin{aligned}
\frac{1}{T}\int_0^T \|(S*h)(t)\|\,dt &\le \frac{1}{T}\int_0^T \left[\int_0^t \|S(t-s)\|\|h(s)\|\,ds\right] dt \\
&= \frac{1}{T}\int_0^T \left[\int_s^T \|S(s)\|\|h(t-s)\|\,dt\right] ds \\
&= \frac{1}{T}\int_0^T \|S(s)\| \left[\int_s^T \|h(t-s)\|\,dt\right] ds \\
&= \int_0^T \|S(s)\| \left[\frac{1}{T}\int_0^{T-s} \|h(u)\|\,du\right] ds \\
&= \int_0^T \|S(s)\|\varphi_T(s)\,ds \\
&\le \int_0^T \|S(s)\|(\varphi_T(0))\,ds \le \varphi_T(0)\int_0^\infty \|S(s)\|\,ds \to 0,\ (\text{as } t\to\infty).
\end{aligned}$$

Hence $S*h \in P_0(\mathbb{R}_+, X)$. □

In order to give an operator theoretical approach to equation (1.1) we have the following definition.

Definition 2.5. *([15]) Let $\mu \ge 0$ and $0 \le \alpha, \beta \le 1$ be given. Let A be a closed linear operator with domain $D(A)$ defined on a Banach space X. We call A the generator of an $(\alpha,\beta)_\mu$-regularized family if there exist $\omega \ge 0$ and a strongly continuous function $S_{\alpha,\beta} : \mathbb{R}_+ \to \mathcal{B}(X)$ such that $\{\lambda^{\alpha+1} + \mu\lambda^\beta : Re\lambda > \omega\} \subset \rho(A)$ and*

$$H(\lambda)x := \lambda^\alpha(\lambda^{\alpha+1} + \mu\lambda^\beta - A)^{-1}x = \int_0^\infty e^{-\lambda t} S_{\alpha,\beta}(t)x\,dt, \quad Re\lambda > \omega, \quad x \in X.$$

Because of the uniqueness theorem for the Laplace transform, if $\mu = 0$ and $\alpha = 0$, this corresponds to the case of a C_0-semigroup whereas the case $\mu = 0,\ \alpha = 1$ corresponds to the concept of cosine family. For more details on the Laplace transform approach to semigroups and cosine functions, we refer to the monograph [2].

Let us recall that a closed and densely defined operator A is said to be ω-sectorial of angle θ if there exists $\theta \in [0, \pi/2)$ and $\omega \in \mathbb{R}$ such that its resolvent exists in the sector

$\omega + S_\theta := \{\omega + \lambda : \lambda \in \mathbb{C}, |arg(\lambda)| < \frac{\pi}{2} + \theta\} \setminus \{\omega\}$, and

$$||(\lambda - A)^{-1}|| \le \frac{M}{|\lambda - \omega|}, \quad \lambda \in \omega + S_\theta. \tag{2.1}$$

These are generators of holomorphic semigroups. In case $\omega = 0$ we merely say that A is sectorial of angle θ. We should mention that in the general theory of sectorial operators, it is not required that (2.1) holds in a sector of angle $\pi/2$. Our restriction corresponds to the class of operators used in this paper.

Sufficient conditions to obtain generators of an $(\alpha,\beta)_\mu$-regularized family are given in the following result.

Theorem 2.6. *([15]) Let $0 < \alpha \le \beta \le 1$, $\mu > 0$ and A be a ω sectorial operator of angle $\beta\pi/2$. Then A generates a bounded $(\alpha,\beta)_\mu$-regularized family.*

We next consider the linear fractional differential equation

$$D_t^{\alpha+1}u(t) + \mu D_t^\beta u(t) - Au(t) = D_t^\alpha f(t), \quad t \ge 0, \quad 0 < \alpha \le \beta \le 1,\ \mu \ge 0, \tag{2.2}$$

with initial conditions $u(0) = x$, $u'(0) = y$ and A is a ω-sectorial operator of angle $\beta\pi/2$.

Recall that a function $u \in C^1(\mathbb{R}_+; X)$ is called a strong solution of (2.2) on $\mathbb{R}_+$ if $u(t) \in D(A)$ and (2.2) holds on $\mathbb{R}_+$. We have the following result.

If A is ω-sectorial of angle $\beta\pi/2$ then, by [15, Cor.3.4] and Theorem 2.6, a strong solution for (2.2) always exists and is given by:

(2.3)

$$u(t) = S_{\alpha,\beta}(t)x + (g_1 * S_{\alpha,\beta})(t)y + \mu(g_{1+\alpha-\beta} * S_{\alpha,\beta}(t))x + (S_{\alpha,\beta} * f)(t), \quad 0 < \alpha \le \beta \le 1,\ \ \mu > 0,$$

where $x, y \in D(A)$; $f : \mathbb{R}_+ \to D(A)$ and $S_{\alpha,\beta}(t)$ is the $(\alpha,\beta)_\mu$-regularized family generated by A. If merely $x, y \in X$ and $f : \mathbb{R}_+ \to X$ instead of the domain of A, we say that u given by the formula (2.3) is a *mild solution* of the linear equation (2.2).

In order to study the pseudo asymptotic behavior of mild solutions, we need the following result on the integrability of the $(\alpha,\beta)_\mu$-regularized family generated by A.

Theorem 2.7. *([15]) Let $0<\alpha\le\beta\le 1$, $\mu>0$ and $\omega<0$. Assume that A is an ω-sectorial operator of angle $\beta\pi/2$, then A generates an $(\alpha,\beta)_\mu$-regularized family $S_{\alpha,\beta}(t)$ satisfying the estimate*

$$||S_{\alpha,\beta}(t)|| \le \frac{C}{1+|\omega|(t^{\alpha+1}+\mu t^{\beta})}, \quad t\ge 0, \tag{2.4}$$

for some constant $C>0$ depending only on α,β.

3. Pseudo asymptotic solutions: The linear case

Let $\mathcal{M}(X)\in\{PP_T(\mathbb{R}_+,X), PAP(\mathbb{R}_+,X), PAA_c(\mathbb{R}_+,X), PAA(\mathbb{R}_+,X)\}$. We can prove the following theorem which is the main result in this section.

Theorem 3.1. *Let $0<\alpha\le\beta\le 1$ and $\mu>0$. Assume that A is an ω-sectorial operator of angle $\beta\pi/2$ with $\omega<0$. Then for each $f\in\mathcal{M}(X)$ there exists a unique mild solution u of equation (2.2) such that $u\in\mathcal{M}(X)$.*

Proof. Let $f\in\mathcal{M}(X)$ be given. By Theorem 2.7, A generates a uniformly integrable $(\alpha,\beta)_\mu$-regularized family $S_{\alpha,\beta}(t)$ on the Banach space X, and the unique mild solution for (2.2) is given by (2.3), that is;

$$u(t)=S_{\alpha,\beta}(t)x+(g_1*S_{\alpha,\beta})(t)y+\mu(g_{1+\alpha-\beta}*S_{\alpha,\beta}(t))x+(S_{\alpha,\beta}*f)(t), \quad 0<\alpha\le\beta\le 1,\ \ \mu>0,$$

where $x,y\in X$. Let $\Lambda\in\{P_T(X), AP(X), AA_c(X), AA(X)\}$. We claim that $S_{\alpha,\beta}*f\in\mathcal{M}(X)$. In fact, for $f=g+h$ where $g\in\Lambda$ and $h\in P_0(\mathbb{R}_+,X)$, we have that

$$(S_{\alpha,\beta}*f)(t)=\int_{-\infty}^{t}S_{\alpha,\beta}(t-s)g(s)ds-\int_{-\infty}^{0}S_{\alpha,\beta}(t-s)g(s)ds+\int_{0}^{t}S_{\alpha,\beta}(t-s)h(s)ds.$$

By [25, Th. 3.3] we conclude that the first term on the right hand side of the above equality belongs to Λ. On other hand, Lemma 2.3 and Lemma 2.4 imply that the second and third term on the right hand side belong to $P_0(\mathbb{R}_+, X)$.

Now, note that by (2.4) we have $\lim_{t\to\infty} \|S_{\alpha,\beta}(t)\| = 0$. From the Lemma 2.2 we obtain that $S_{\alpha,\beta}(t) \in P_0(\mathbb{R}_+, X)$. Hence $S_{\alpha,\beta} \in \mathcal{M}(X)$. We now prove that $g_1 * S_{\alpha,\beta} \in \mathcal{M}(X)$. In fact, by (2.4) we have $\sup_{t>\tau} ||tS_{\alpha,\beta}(t)|| < \infty$, for each $\tau > 0$. Since A is an ω-sectorial of angle $\beta\frac{\pi}{2}$ then $||\hat{S}_{\alpha,\beta}(\lambda)|| \to 0$ as $\lambda \to 0$. Thus, by the vector-valued Hardy-Littlewood theorem (see [2, Theorem 4.2.9]) we conclude that $||(g_1 * S_{\alpha,\beta})(t)|| \to 0$ as $t \to \infty$. The conclusion follows from Lemma 2.2. It remains only to show that $g_{1+\alpha-\beta} * S_{\alpha,\beta} \in \mathcal{M}(X)$ for $\alpha < \beta$. To see this, we estimate $||g_{1+\alpha-\beta} * S_{\alpha,\beta}(t)||$ as follows. Let $0 < \epsilon < \beta - \alpha$ be given, then

$$\begin{aligned} ||g_{1+\alpha-\beta} * S_{\alpha,\beta}(t)|| &= ||\Gamma(\beta-\alpha-\epsilon)\int_0^t g_{1+\alpha-\beta}(t-\tau)g_{\beta-\alpha-\epsilon}(\tau)\tau^{\alpha-\beta+\epsilon+1}S_{\alpha,\beta}(\tau)d\tau|| \\ &\le \Gamma(\beta-\alpha-\epsilon)\int_0^t g_{1+\alpha-\beta}(t-\tau)g_{\beta-\alpha-\epsilon}(\tau)\tau^{\alpha-\beta+\epsilon+1}||S_{\alpha,\beta}(\tau)||d\tau \end{aligned}$$

where, thanks to (2.4), we have that

$$\Gamma(\beta-\alpha-\epsilon)\tau^{\alpha-\beta+\epsilon+1}||S_{\alpha,\beta}(\tau)|| \le \frac{M\tau^{\alpha-\beta+\epsilon-1}}{1+|\omega|\tau^{\alpha+1}} = \frac{M\tau^{-\beta+\epsilon}}{\frac{1}{\tau^{\alpha+1}}+|\omega|}, \quad \tau > 0.$$

Since $\epsilon < \beta$, there exists a constant $C > 0$ such that $\tau^{\alpha-\beta+\epsilon+1}||S_{\alpha,\beta}(\tau)|| \le C$. Therefore,

$$||g_{1+\alpha-\beta} * S_{\alpha,\beta}(t)|| \le C\int_0^t g_{1+\alpha-\beta}(t-\tau)g_{\beta-\alpha-\epsilon}(\tau)d\tau = Cg_{1-\epsilon}(t) = Ct^{-\epsilon},$$

which shows that $\|g_{1+\alpha-\beta} * S_{\alpha,\beta}(t)\| \to 0$ as $t \to \infty$. By Lemma 2.2 we can conclude that $g_{1+\alpha-\beta} * S_{\alpha,\beta}(t) \in P_0(\mathbb{R}_+, X)$. Therefore $g_{1+\alpha-\beta} * S_{\alpha,\beta} \in \mathcal{M}(X)$ and finally, we have shown that $u \in \mathcal{M}(X)$. □

For further use, we state the following immediate corollaries. The first, shows existence and uniqueness of pseudo almost periodic mild solutions of equation (2.2).

Corollary 3.2. *Let $0 < \alpha \leq \beta \leq 1$ and $\mu > 0$. Assume that A is an ω-sectorial operator of angle $\beta\pi/2$ with $\omega < 0$. Then for each $f \in PAP(\mathbb{R}_+, X)$ there exists a unique mild solution u of equation (2.2) such that $u \in PAP(\mathbb{R}_+, X)$.*

We next give existence and uniqueness of pseudo almost automorphic mild solutions of equation (2.2).

Corollary 3.3. *Let $0 < \alpha \leq \beta \leq 1$ and $\mu > 0$. Assume that A is an ω-sectorial operator of angle $\beta\pi/2$ with $\omega < 0$. Then for each $f \in PAA(\mathbb{R}_+, X)$ there exists a unique mild solution u of equation (2.2) such that $u \in PAA(\mathbb{R}_+, X)$.*

4. Pseudo Asymptotic solutions: The semilinear case

Define the Nemytskii superposition operator $\mathcal{N}(\varphi)(\cdot) := f(\cdot, \varphi(\cdot))$ for $\varphi \in \mathcal{M}(X)$. We define the set $\mathcal{M}(\mathbb{R}_+ \times X; X)$ to consist of all functions $f : \mathbb{R}_+ \times X \to X$ such that $f(\cdot, x) \in \mathcal{M}(X)$ uniformly for each $x \in K$, where K is any bounded subset of X. From now on, we also denote

$$P_0(\mathbb{R}_+ \times X, X) = \{f \in BC(\mathbb{R}_+ \times X, X) : \lim_{T\to\infty} \frac{1}{T}\int_0^T ||f(t,x)||dt = 0 \text{ uniformly on any subset of } X\}.$$

In what follows we study existence and uniqueness of solutions in $\mathcal{M}(X)$ for the semi-linear fractional order differential equation

$$D_t^{\alpha+1}u(t) + \mu D_t^{\beta}u(t) - Au(t) = D_t^{\alpha}f(t, u(t)), \quad t \geq 0, \quad 0 < \alpha \leq \beta \leq 1, \quad \mu > 0, \tag{4.1}$$

where A is an ω-sectorial operator of angle $\beta\pi/2$ with $\omega < 0$, $u(0) = x$ and $u'(0) = y$.

In view of the linear case, the following definition of mild solution is natural. Note that in the borderline case $\mu = 0$ and $\alpha = 1$ it corresponds to the notion of mild solution for the semi-linear problem $u''(t) = Au(t) + f(t, u(t))$ under the hypothesis that A is the generator of a cosine family $C(t)$. In fact, in this case: $S_{1,0}(t) \equiv C(t)$ and the associate sine family is equal to $(g_1 * S_{1,0})(t)$.

Definition 4.1. *Suppose* $0 < \alpha \le \beta \le 1$, $\mu > 0$. *A function* $u : \mathbb{R}_+ \to X$ *is said to be a mild solution to Equation (4.1) if it satisfies*

$$u(t) = S_{\alpha,\beta}(t)x + (g_1 * S_{\alpha,\beta})(t)y + \mu(g_{1+\alpha-\beta} * S_{\alpha,\beta}(t))x + \int_0^t S_{\alpha,\beta}(t-s)f(s,u(s))ds, \tag{4.2}$$

for each $t \in \mathbb{R}_+$ *and* $x, y \in X$.

We next give a result on existence of mild solutions for the semi-linear problem.

Theorem 4.2. *Let* $0 < \alpha \le \beta \le 1$ *and* $\mu > 0$. *Assume that* A *is an* ω*-sectorial operator of angle* $\beta\pi/2$ *and* $\omega < 0$. *Let* $f : \mathbb{R}_+ \times X \to X$ *be a function on* $\mathcal{M}(\mathbb{R}_+ \times X; X)$ *and assume that there exists a bounded integrable function* $L_f : \mathbb{R}_+ \to \mathbb{R}_+$ *satisfying*

$$||f(t,x) - f(t,y)|| \le L_f(t)||x - y||, \tag{4.3}$$

for all $x, y \in X$ *and* $t \ge 0$. *Then Equation (4.1) has a unique mild solution* $u \in \mathcal{M}(X)$.

Proof. Let $S_{\alpha,\beta}(t)$ be the $(\alpha,\beta)_\mu$-regularized family generated by A (cf. Theorem 2.7). We define the operator $K_{\alpha,\beta}$ on the space $\mathcal{M}(X)$ by

$$(K_{\alpha,\beta}u)(t) = S_{\alpha,\beta}(t)x + (g_1 * S_{\alpha,\beta})(t)y + \mu(g_{1+\alpha-\beta} * S_{\alpha,\beta}(t))x + \int_0^t S_{\alpha,\beta}(t-s)f(s,u(s))ds. \tag{4.4}$$

From the proof of Theorem 3.1, we know that $S_{\alpha,\beta}(t)x + (g_1 * S_{\alpha,\beta})(t)y + \mu(g_{1+\alpha-\beta} * S_{\alpha,\beta}(t))x \in \mathcal{M}(X)$. Moreover by [25, Theorem 4.1] we conclude that the function $s \to f(s,u(s))$ is in $\mathcal{M}(X)$. Then, by hypothesis and in the same way as in the proof of Theorem 3.1, we arrive at the conclusion that $\int_0^t S_{\alpha,\beta}(t-s)f(s,u(s))ds$ is also in $\mathcal{M}(X)$

and thus $K_{\alpha,\beta}$ is well defined. Let u, v be in $\mathcal{M}(X)$. Observe that

$$\begin{aligned}||(K_{\alpha,\beta}u)(t)-(K_{\alpha,\beta}v)(t)|| &\leq \int_0^t \|S_{\alpha,\beta}(t-s)\|\|f(s,u(s))-f(s,v(s))\|ds \\ &\leq \int_0^t \|S_{\alpha,\beta}(t-s)\|L_f(s)\|u(s)-v(s)\|ds \\ &\leq \|S_{\alpha,\beta}\|_1\|u-v\|_\infty \int_0^t L_f(s)ds \leq \|S_{\alpha,\beta}\|_1\|u-v\|_\infty\|L_f\|_1.\end{aligned}$$

By induction, we find the following estimate:

$$\begin{aligned}||(K^n_{\alpha,\beta}u)(t)-(K^n_{\alpha,\beta}v)(t)|| &\leq \frac{||S_{\alpha,\beta}||_1^n}{(n-1)!}||u-v||_\infty \int_0^t L_f(s)\left(\int_0^s L_f(\tau)d\tau\right)^{n-1} ds \\ &= \frac{||S_{\alpha,\beta}||_1^n}{n!}||u-v||_\infty \left(\int_0^t L_f(\tau)d\tau\right)^n \\ &\leq \frac{||S_{\alpha,\beta}||_1^n}{n!}||u-v||_\infty ||L_f||_1^n.\end{aligned}$$

Since $\frac{||S_{\alpha,\beta}||_1^n}{n!}||L_f||_1^n < 1$ for n sufficiently large, applying the contraction principle we conclude that F has a unique fixed point $u \in \mathcal{M}(X)$ such that $(K_{\alpha,\beta}u)(t) = u(t)$. $\square$

The following corollaries are immediate consequences.

Corollary 4.3. *Let $0 < \alpha \leq \beta \leq 1$ and $\mu > 0$. Assume that A is an ω-sectorial operator of angle $\beta\pi/2$ and $\omega < 0$. Let $f : \mathbb{R}_+ \times X \to X$ be a function on $PAP(\mathbb{R}_+ \times X; X)$ and assume that there exists a bounded integrable function $L_f : \mathbb{R}_+ \to \mathbb{R}_+$ satisfying (4.3). Then Equation (4.1) has a unique mild solution $u \in PAP(\mathbb{R}_+, X)$.*

Corollary 4.4. *Let $0 < \alpha \leq \beta \leq 1$ and $\mu > 0$. Assume that A is an ω-sectorial operator of angle $\beta\pi/2$ and $\omega < 0$. Let $f : \mathbb{R}_+ \times X \to X$ be a function on $PAA(\mathbb{R}_+ \times X; X)$ and assume that there exists a bounded integrable function $L_f : \mathbb{R}_+ \to \mathbb{R}_+$ satisfying (4.3). Then Equation (4.1) has a unique mild solution $u \in PAA(\mathbb{R}_+, X)$.*

To finish, we present one example, which do not aim at generality but indicate how our theorems can be applied to concrete problems.

Example 4.5. *Suppose that* $b \in L^1(\mathbb{R}_+)$ *and* $b(t) \to 0$ *as* $t \to \infty$. *Then the equation*

(4.5)

$$D_t^{\alpha+1}u(x,t)+\mu D_t^{\beta}u(x,t) = \frac{\partial^2}{\partial x^2}u(x,t)+\tau u(x,t)+D_t^{\alpha}[b(t)\sin(u(t))], \quad t>0, \quad 0<\alpha\leq\beta\leq 1,$$

where $\tau < 0$ *is fixed, with initial and zero boundary conditions has a unique mild solution* $u(t,x)$ *which decomposes as a sum of a first part which is almost automorphic (possibly zero) and a second part that belongs to the space* $P_0(\mathbb{R}_+ \times X, X)$.

Indeed, the equation (4.5) is of the form (4.1) with $Au = \frac{\partial^2}{\partial x^2}u + \tau u$ and $f(t,u) = b(t)\sin(u(t))$. Setting the Dirichlet boundary conditions $u(0,t) = u(2\pi,t) = 0$ we consider A with domain $D(A) := \{u \in L^2[0,2\pi] : u'' \in L^2[0,2\pi]; u(0) = u(2\pi) = 0\}$ and $f(t,x) = b(t)\sin(x)$. Then it is wellknown that the operator A is ω sectorial with $\omega = \tau < 0$ and angle $\pi/2$ (and hence of angle $\beta\pi/2$ for all $\beta \leq 1$). On the other hand, since $b \in L^1(\mathbb{R}_+)$ and $b(t) \to 0$ as $t \to \infty$, we have

$$\|f(t,u) - f(t,v)\|_2^2 = \int_0^{\pi} |b(t)|^2|\sin(u(s)) - \sin(v(s))|^2 ds \leq |b(t)|^2\|u-v\|_2^2,$$

and the condition (4.3) holds. Hence the hypothesis of Theorem 4.2 are satisfied and thus the conclusion of the example follows.

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