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WEIGHTED PSEUDO ALMOST AUTOMORPHIC AND $\mathcal{S}$-ASYMPTOTICALLY ω-PERIODIC SOLUTIONS TO FRACTIONAL DIFFERENCE-DIFFERENTIAL EQUATIONS

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Abstract. We study weighted pseudo almost automorphic solutions for the nonlinear fractional difference equation

$$\Delta^{\alpha}u(n) = Au(n+1) + f(n, u(n)), \quad n \in \mathbb{Z},$$

for $0 < \alpha \leq 1$, where A is the generator of an α-resolvent sequence $\{S_\alpha(n)\}_{n\in\mathbb{N}_0}$ in $\mathcal{B}(X)$. We prove the existence and uniqueness of a weighted pseudo almost automorphic solution assuming that $f(\cdot,\cdot)$ is weighted almost automorphic in the first variable and satisfies a Lipschitz (local and global) type condition in the second variable. An analogous result is also proved for $\mathcal{S}$-asymptotically ω-periodic solutions.

1. Introduction

In this article, we study sufficient conditions for the existence and uniqueness of discrete weighted pseudo almost automorphic solutions to the semilinear fractional difference - differential equation

$$\Delta^{\alpha}u(n) = Au(n+1) + f(n, u(n)), \quad n \in \mathbb{Z}, \tag{1.1}$$

where $0 < \alpha \leq 1$, A is a closed linear operator with domain $D(A)$ defined on a Banach space X which generates an α-resolvent sequence $\{S_\alpha(n)\}_{n\in\mathbb{N}_0} \subset \mathcal{B}(X)$ and $f : \mathbb{Z} \times X \to X$ is a discrete weighted pseudo almost automorphic function in $k \in \mathbb{Z}$ satisfying suitable Lipschitz type conditions with respect to $x \in X$. The fractional difference is understood in the sense defined in [1], which is analogous to the Weyl fractional derivative in the continuous case. See Definition 2.10 below. Difference-differential equations appear in many practical situations, for instance in traffic dynamics, theory of probability, theory of chain processes of chemistry, radioactivity and in biological models, see e.g. [9, 10, 11]. Nonlinear difference equations has been studied by several authors, see e.g. [5, 8, 14, 18, 19, 23, 24]. First studies on extensions of the notion of almost automorphic sequences are due to Fink [17]. The concept of discrete weighted pseudo almost automorphic functions was introducing by Abbas [2] in 2010 as a further generalization of almost automorphic sequences. Agarwal et al. [7] obtained almost automorphic solutions to a nonlinear Volterra difference equation. Ding et al. [16] studied the weighted pseudo almost periodic

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solutions for a class of discrete hematopoiesis model. Xia [29] extended the space of weighted pseudo almost automorphic sequences with the help of two weights, proved fundamental properties of this type of functions and apply his results to Volterra difference equations. Then, Li and Sun proved applications to some semi-linear difference equations [22]. Abbas et al. defined the concept of Stepanov type weighted pseudo almost automorphic sequences and proved an important composition theorem [3]. An interesting application of almost automorphic sequences to a model of a cellular neural network was shown by Abbas [4]. For a recent application to discrete delayed neutral systems, see [6]. For other developments, see [12].

Recently, Abadias and Lizama [1] proved the existence and uniqueness of almost automorphic solutions for (1.1), where A is the generator of a C_0-semigroup. There the authors introduced the concept of fractional difference in the Weyl-like sense and the notion of α-resolvent sequence.

Our motivation for this article stems from the fact that equations of type (1.1) can arise in many problems of science and engineering either directly or as part of a discretization process [20, 21] and that the study of weighted pseudo almost automorphic solutions for fractional difference-differential equations does not exist at this time. Since the qualitative behavior of the solutions is crucial in order to better understand the underlying structure, the existence of weighted pseudo almost automorphic solutions for such equations seems to be highly important. We observe that the research on these properties for fractional difference equations is in its early stages. For some limited results we refer the reader in particular to [23], [24] and [25].

In this article we prove the following new results: Let $0 < \alpha < 1$ and A be the generator of an exponentially stable C_0-semigroup $T(t)$ with growth bound $\omega_0(A)$. Suppose that $f : \mathbb{Z} \times X \to X$ can be decomposed as $f = g + \varphi$ where f is discrete almost automorphic and φ is weight mean ergodic. Assume also that f is globally Lipschitz with constant satisfying the estimate $L < \frac{1}{\omega_0(A)}$. Then, there exists a unique solution in a mild sense, that can also be decomposed as $u = v + \nu$ where v and ν have the same regularity as g and φ, respectively. See Corollary 3.3 for a precise description of this result. Using the same methods, we obtain an analogous result on the regularity of solutions for those data f with the property that there exists an integer number ω such that $f(n+\omega)-f(n)$ goes to zero in weighted mean. Nonlocal versions of this results are also provided. See Theorem 3.4 and Corollary 3.5. In order to prove this kind of results, we introduce a new convolution theorem, which provides regularity for the linearized version of (1.1). See Theorem 2.13.

This article is organized as follows: Section 2 is devoted to preliminaries, where we prove a convolution theorem which is new in the context of α-resolvent families associated with the operator that appears in (1.1). In Section 3 we present our main results on the existence and uniqueness of weighted pseudo almost automorphic and $\mathcal{S}$-asymptotically ω-periodic solutions. Finally, in Section 4 we give a concrete example to illustrate the main findings.

2. Preliminaries

In this section, we give the basic definitions and essential results that we will be used later. We first introduce the following spaces of sequences.

(i) $s(\mathbb{Z}, X)$ is the vector space of all vector valued sequences.
(ii) $BS(\mathbb{Z}, X) := \{f : \mathbb{Z} \to X : \|f\|_\infty := \sup_{n\in\mathbb{Z}} \|f(n)\| < \infty\}$.

(iii) $l^p_\rho(\mathbb{Z},X) := \{f : \mathbb{Z} \to X : \|f\|_{l^p_\rho} := \sum_{n=-\infty}^{\infty} \|f(n)\|^p \rho(n) < \infty\}$ where $\rho : \mathbb{Z} \to (0,\infty)$ is a positive sequence.
(iv) $C_0(\mathbb{Z},X) := \{f \in BS(\mathbb{Z},X) : \lim_{n\to\infty} \|f(n)\| = 0\}$.
(v) $C_\omega(\mathbb{Z},X) = \{f \in BS(\mathbb{Z},X) : f$ is ω-periodic$\}$.
(vi) $\mathfrak{UC}(\mathbb{Z}\times X,X)$ is the set of all functions $f : \mathbb{Z}\times X \to X$ satisfying that for all $\epsilon > 0$ there exists $\delta > 0$ such that

$$\|f(k,x) - f(k,y)\| \le \epsilon$$

for all $k \in \mathbb{Z}$ and for all $x, y \in X$ with $\|x - y\| < \delta$.

Let U be the set of all sequences $\rho : \mathbb{Z} \to (0,\infty)$ which are locally summable over $\mathbb{Z}$. For a given $\rho \in U$ and $K \in \mathbb{Z}_+$ we denote

$$m_d(K,\rho) = \sum_{k=-K}^{K} \rho(k).$$

Define

$$U_\infty = \{\rho \in U : \lim_{K\to\infty} m_d(K,\rho) = \infty\},$$
$$U_b = \{\rho \in U_\infty : 0 < \inf_{k\in\mathbb{Z}} \rho(k) \le \sup_{k\in\mathbb{Z}} \rho(k) < \infty\}.$$

We have that $U_b \subset U_\infty \subset U$.

Let $\rho_1, \rho_2 \in U_\infty$. ρ_1 is said to be equivalent to ρ_2 if $\rho_1/\rho_2 \in U_b$. In this case we write $\rho_1 \sim \rho_2$. It can be proved that $U_\infty = \cup_{\rho\in U_\infty}\{\varrho \in U_\infty : \rho \sim \varrho\}$.

Let $\rho \in U_\infty$ and $m \in \mathbb{Z}$. We define $\rho_m(n) = \rho(n+m)$ for $n \in Z$ and

$$U_T = \{\rho \in U_\infty : \rho \sim \rho_m \text{ for each } m \in \mathbb{Z}\}.$$

A sequence $f : \mathbb{Z} \to X$ is called almost automorphic if for every integer sequence $\{k_n'\}$, there exists a subsequence $\{k_n\}$ such that

$$\bar{f}(k) := \lim_{n\to\infty} f(k + k_n)$$

is well defined for each $k \in \mathbb{Z}$ and $\lim_{n\to\infty} \bar{f}(k - k_n) = f(k)$, see [8, Definition 2.1] and references therein. We denote by $AA_d(\mathbb{Z},X)$ the set of almost automorphic sequences. It is well known that the set $AA_d(\mathbb{Z},X)$ endowed with the norm $\|f\|_\infty := \sup_{k\in\mathbb{Z}} \|f(k)\|$ is a Banach space, see [8, Theorem 2.4]. A typical example is $f(k) = \sin\big(\frac{1}{2+\cos(k)+\cos(\sqrt{2}k)}\big)$, $k \in \mathbb{Z}$. A sequence $f : \mathbb{Z}\times X \to X$ is said to be almost automorphic if $f(k,x)$ is almost automorphic in $k \in \mathbb{Z}$ for any $x \in X$. We denote this space by $AA_d(\mathbb{Z}\times X,X)$.

Let $\rho_1 \in U_\infty$. Define the ergodic space (see [2]) by

$$PAA_0S(\mathbb{Z},X,\rho_1) = \Big\{f \in BS(\mathbb{Z},X) : \lim_{K\to\infty} \frac{1}{m_d(K,\rho_1)} \sum_{k=-K}^{K} \|f(k)\|\rho_1(k) = 0\Big\}.$$

Particularly, for $\rho_1, \rho_2 \in U_\infty$ (see [29]),

$$PAA_0S(\mathbb{Z},X,\rho_1,\rho_2) = \Big\{f \in BS(\mathbb{Z},X) : \lim_{K\to\infty} \frac{1}{m_d(K,\rho_1)} \sum_{k=-K}^{K} \|f(k)\|\rho_2(k) = 0\Big\}.$$

Remark 2.1. Note that if $\rho_1 \sim \rho_2$ then

$$PAA_0S(\mathbb{Z},X,\rho_1,\rho_2) = PAA_0S(\mathbb{Z},X,\rho_1) = PAA_0S(\mathbb{Z},X,\rho_2).$$

Let $\rho_1, \rho_2 \in U_\infty$. A sequence $f : \mathbb{Z} \to X$ is called discrete weighted pseudo almost automorphic if it can be expressed as $f = g + \varphi$, where $g \in AA_d(\mathbb{Z}, X)$ and $\varphi \in PAA_0S(\mathbb{Z}, X, \rho_1, \rho_2)$, see [29, Definition 8] and references therein. The set of such functions is denoted by $WPAA_d(\mathbb{Z}, X)$. It is well known that the set $WPAA_d(\mathbb{Z}, X)$ is a Banach space with the norm $\|f\|_\infty = \sup_{k\in\mathbb{Z}} \|f(k)\|$ (see [29, Lemma 10]). A classical example is the function $f(k) = \text{signum}(\cos 2\pi k\theta) + e^{-|k|}$ with $\rho_1(k) = \rho_2(k) = 1 + k^2$ for $k \in \mathbb{Z}$ (see [2]).

Remark 2.2. If $\rho_1 \sim \rho_2$, then $WPAA_d(\mathbb{Z}, X)$ coincide with the discrete weighted pseudo almost automorphic functions $WPAAS(\mathbb{Z})$ defined in [2].

Similarly, we define (see [29])

$$
\begin{aligned}
&PAA_0S(\mathbb{Z} \times X, X, \rho_1, \rho_2)\\
&= \Big\{ f \in BS(\mathbb{Z} \times X, X) : \lim_{K\to\infty} \frac{1}{m_d(K, \rho_1)} \sum_{k=-K}^{K} \|f(k, x)\|\rho_2(k) = 0\\
&\quad \text{uniform in } x \in X \Big\}.
\end{aligned}
$$

A function $f : \mathbb{Z}\times X \to X$ is said to be discrete weighted pseudo almost automorphic in $k \in \mathbb{Z}$ for each $x \in X$, if it can be decomposed as $f = g+\varphi$, where $g \in AA_d(\mathbb{Z}\times X)$ and $\varphi \in PAA_0S(\mathbb{Z} \times X, X, \rho_1, \rho_2)$. Denote by $WPAA_d(\mathbb{Z} \times X, X)$ the set of such functions.

Throughout the rest of this paper, we denote by V_∞ the set of all functions $\rho_1, \rho_2 \in U_\infty$ satisfying that there exists an unbounded set $\Omega \subset \mathbb{Z}$ such for all $m \in \mathbb{Z}$,

$$
\lim_{|k|\to\infty, k\in\Omega} \sup \frac{\rho_2(k + m)}{\rho_1(k)} < \infty,
$$
$$
\lim_{K\to\infty} \frac{\sum_{k\in([-K,K]\setminus\Omega)+m} \rho_2(k)}{m_d(K, \rho_1)} = 0.
$$

Xia [29] proved the following composition theorem.

Theorem 2.3 ([29, Th. 16])**.** *Assume that $\rho_1, \rho_2 \in V_\infty$, and that $f \in WPAA_d(\mathbb{Z}\times X, X) \cap \mathfrak{UC}(\mathbb{Z} \times X, X)$ and $h \in WPAA_d(\mathbb{Z}, X)$. Then $f(\cdot, h(\cdot)) \in WPAA_d(\mathbb{Z}, X)$.*

We recall that a function $f : \mathbb{Z} \times X \to X$ is said to be locally Lipschitz with respect to the second variable if for each positive number r, for all $k \in \mathbb{Z}$ and for all $x, y \in X$ with $\|x\| \le r$ and $\|y\| \le r$, we have $\|f(k, x) - f(k, y)\| \le L(r)\|x - y\|$, where $L : \mathbb{R}_+ \to \mathbb{R}_+$ is a nondecreasing function.

The previous theorem admits a new version with local conditions on the function f.

Corollary 2.4. *Let $\rho_1, \rho_2 \in V_\infty$. Let $f : \mathbb{Z} \times X \to X$ be a discrete weighted pseudo almost automorphic function in the first variable and locally Lipschitz in the second variable. Then the conclusion of the previous theorem holds.*

A function $f \in BS(\mathbb{Z}, X)$ is called discrete asymptotically ω-periodic if there exist $g \in C_\omega(\mathbb{Z}, X)$, $\varphi \in C_0(\mathbb{Z}, X)$ such that $f = g + \varphi$. The collection of such functions is denoted by $AP_\omega(\mathbb{Z}, X)$. A function $f \in BS(\mathbb{Z}, X)$ is called discrete $\mathcal{S}$-asymptotic ω-periodic if there exists $\omega \in \mathbb{Z}^+ \setminus \{0\}$ such that $\lim_{n\to\infty}(f(n + \omega) - f(n)) = 0$, see [30, Definition 5] and references therein. The collection of such functions is denoted by $\mathcal{S}AP_\omega(\mathbb{Z}, X)$.

Let $\rho \in U_\infty$. A function $f \in BS(\mathbb{Z}, X)$ is called discrete pseudo-$\mathcal{S}$-asymptotic ω-periodic if there exists $\omega \in \mathbb{Z}^+ \setminus \{0\}$ such that

$$\lim_{n\to\infty} \frac{1}{2n} \sum_{k=-n}^{n} \|f(k+\omega) - f(k)\| = 0;$$

see [30, Definition 6]. The collection of such functions is denoted by $P\mathcal{S}AP_\omega(\mathbb{Z}, X)$.

Let $\rho \in U_\infty$. A function $f \in BS(\mathbb{Z}, X)$ is called discrete weighted pseudo-$\mathcal{S}$-asymptotic ω-periodic if there exists $\omega \in \mathbb{Z}^+ \setminus \{0\}$ such that

$$\lim_{n\to\infty} \frac{1}{m_d(n,\rho)} \sum_{k=-n}^{n} \|f(k+\omega) - f(k)\|\rho(k) = 0,$$

see [30, Definition 7]. The set of such functions is denoted by $WP\mathcal{S}AP_\omega(\mathbb{Z}, X, \rho)$. It is clear that $AP_\omega(\mathbb{Z}, X) \subset P\mathcal{S}AP_\omega(\mathbb{Z}, X) \subset WP\mathcal{S}AP_\omega(\mathbb{Z}, X, \rho)$. It is well known that the set $WP\mathcal{S}AP_\omega(\mathbb{Z}, X, \rho)$ is a Banach space with the norm $\|f\|_\infty := \sup_{k\in\mathbb{Z}} \|f(k)\|$ (see [30, Lemma 8]).

Remark 2.5 ([30]). *If $\rho_1, \rho_2 \in U_\infty$ and $\rho_1 \sim \rho_2$ then*

$$WP\mathcal{S}AP_\omega(\mathbb{Z}, X, \rho_1) = WP\mathcal{S}AP_\omega(\mathbb{Z}, X, \rho_2),$$
$$WP\mathcal{S}AP_\omega(\mathbb{Z}, X, \rho_1/\rho_2) = P\mathcal{S}AP_\omega(\mathbb{Z}, X).$$

Theorem 2.6 ([30, Th. 12]). *Assume that $\rho \in U_\infty$ and that $f \in WP\mathcal{S}AP_\omega(\mathbb{Z} \times X, X, \rho) \cap \mathfrak{UC}(\mathbb{Z} \times X, X)$ and $h \in WP\mathcal{S}AP_\omega(\mathbb{Z}, X, \rho)$. Then*

$$f(\cdot, h(\cdot)) \in WP\mathcal{S}AP_\omega(\mathbb{Z}, X, \rho).$$

Now, we present an alternative version of the preceding theorem with local conditions on f.

Corollary 2.7. *Let $\rho \in U_\infty$. Let $f : \mathbb{Z} \times X \to X$ be a discrete $\mathcal{S}$-asymptotic ω-periodic function in the first variable and locally Lipschitz in the second variable. Then, the conclusion of the previous theorem is true.*

The following definition of discrete derivative in Weyl sense is due to Abadias and Lizama [1]. We define the forward Euler operator $\Delta : s(\mathbb{Z}, X) \to s(\mathbb{Z}, X)$ by

$$\Delta f(n) = f(n+1) - f(n), \quad n \in \mathbb{Z}.$$

Recursively we define

$$\Delta^{k+1} = \Delta^k \Delta = \Delta\Delta^k, \quad k \in \mathbb{N},$$

and $\Delta^0 = I$ is the identity operator. It is easy to see that

$$\Delta^k f(n) = \sum_{j=0}^{k} (-1)^{k-j} \binom{k}{j} f(n+j).$$

In particular $\Delta^1 = \Delta$. In addition, for $\alpha > 0$, we consider the scalar sequence $\{k^\alpha(n)\}_{n\in\mathbb{N}_0}$ defined by

$$k^\alpha(n) := \frac{\Gamma(n+\alpha)}{\Gamma(\alpha)\Gamma(n+1)}.$$

We note that the kernel k^α satisfies the semigroup property in $\mathbb{N}_0$, that is,

$$(k^\alpha * k^\beta)(n) = \sum_{j=0}^{n} k^\alpha(n-j) k^\beta(j) = k^{\alpha+\beta}(n)$$

with $n \in \mathbb{N}_0$ and $\alpha, \beta > 0$.

Definition 2.8. Let $\alpha > 0$ be given and $\rho(n) = |n|^{\alpha-1}, n \in \mathbb{Z}$. The α-th fractional sum of a sequence $f \in l^1_\rho(\mathbb{Z}, X)$ is defined by

$$\Delta^{-\alpha} f(n) := \sum_{j=-\infty}^{n} k^\alpha(n-j) f(j), \quad n \in \mathbb{Z}.$$

See also [23] for related work on a slight variant of this definition.

Remark 2.9. The previous definition can be numerically compared with the continuous fractional integral in the sense of Weyl, see [26, Section 3.3]. Moreover, we observe that as a consequence of the semigroup property of the kernel k^α we have that $\Delta^{-\alpha}\Delta^{-\beta} = \Delta^{-(\alpha+\beta)} = \Delta^{-\beta}\Delta^{-\alpha}$.

Definition 2.10. Let $\alpha > 0$ be given and $\rho(n) = |n|^{\alpha-1}$ for $n \in \mathbb{Z}$. The α-th fractional difference of a sequence $f \in l^1_\rho(\mathbb{Z}, X)$ is defined by

$$\Delta^\alpha f(n) := \Delta^m \Delta^{-(m-\alpha)} f(n), \quad n \in \mathbb{Z},$$

with $m = [\alpha] + 1$.

A sequence $\{S(n)\}_{n\in\mathbb{N}_0} \subset \mathcal{B}(X)$ is called summable if $\|S\|_1 := \sum_{n=0}^{\infty} \|S(n)\| < \infty$. The following definition is introduced in [23].

Definition 2.11. Let $\alpha > 0$ and A be a closed linear operator with domain $D(A)$ defined on a Banach space X. An operator-valued sequence $\{S_\alpha(n)\}_{n\in\mathbb{N}_0} \subset \mathcal{B}(X)$ is called a discrete α-resolvent family generated by A if it satisfies the following conditions

(i) $S_\alpha(n)Ax = AS_\alpha(n)x$ for $n \in \mathbb{N}_0$ and $x \in D(A)$;
(ii) $S_\alpha(n)x = k^\alpha(n)x + A(k^\alpha * S_\alpha)(n)x$, for all $n \in \mathbb{N}_0$ and $x \in X$.

We recall the following practical criteria for summability of α-resolvent families.

Theorem 2.12 ([1, Th. 3.5])**.** *Let $0 < \alpha < 1$ and A be the generator of an exponentially stable C_0-semigroup $\{T(t)\}_{t\geq 0}$ defined on a Banach space X. Then A generates a discrete α-resolvent family $\{S_\alpha(n)\}_{n\in\mathbb{N}_0}$ defined by*

$$S_\alpha(n)x := \int_0^\infty \int_0^\infty e^{-t} \frac{t^n}{n!} f_{s,\alpha}(t) T(s)x \, ds \, dt, \quad n \in \mathbb{N}_0, \ x \in X. \tag{2.1}$$

Moreover, $\{S_\alpha(n)\}_{n\in\mathbb{N}_0}$ is summable. (Here $f_{s,\alpha}(t)$ is the function called stable Lévy process, see [1]*).*

Let $\mathcal{M}(\mathbb{Z}, X) := \{WPAA_d(\mathbb{Z}, X), WPSAP_\omega(\mathbb{Z}, X, \rho)\}$ and $\mathcal{M}(\mathbb{Z} \times X, X) := \{WPAA_d(\mathbb{Z} \times X, X), WPSAP_\omega(\mathbb{Z} \times X, X, \rho)\}$. The following is our main result on regularity under convolution of the above mentioned spaces.

Theorem 2.13. *Let $0 < \alpha < 1$, $\rho_1, \rho_2 \in V_\infty$ and $\rho \in U_T$. Assume that A generates a summable discrete α-resolvent family $\{S_\alpha(n)\}_{n\in\mathbb{N}_0} \subset \mathcal{B}(X)$. If f belongs to one of the spaces $\Omega \in \mathcal{M}(\mathbb{Z}, X)$ then*

$$u(n) = \sum_{j=-\infty}^{n-1} S_\alpha(n-1-j) f(j) \quad (n \in \mathbb{Z})$$

belongs to the same space Ω.

Proof. First note that u is well defined since $\|u(n)\| \leq \|S_\alpha\|_1 \|f\|_\infty$, for all $n \in \mathbb{Z}$. First, we consider $f \in WPAA_d(\mathbb{Z}, X)$. Let $u = f_1 + f_2$, where $f_1 \in AA_d(\mathbb{Z}, X)$ and $f_2 \in PAA_0S(\mathbb{Z}, X, \rho_1, \rho_2)$. Then

$$u(n) = \sum_{j=-\infty}^{n-1} S_\alpha(n-1-j) f_1(j) + \sum_{j=-\infty}^{n-1} S_\alpha(n-1-j) f_2(j) =: u_1(n) + u_2(n).$$

It follows from [1, Theorem 4.5] that $u_1 \in AA_d(\mathbb{Z}, X)$. It remains to prove that $u_2 \in PAA_0S(\mathbb{Z}, X, \rho_1, \rho_2)$. Indeed,

$$\begin{aligned} &\frac{1}{m_d(K, \rho_1)} \sum_{k=-K}^{K} \|u_2(k)\| \rho_2(k) \\ &= \frac{1}{m_d(K, \rho_1)} \sum_{k=-K}^{K} \Big\| \sum_{j=-\infty}^{k-1} S_\alpha(k-1-j) f_2(j) \Big\| \rho_2(k) \\ &\leq \sum_{m=0}^{\infty} \|S_\alpha(m)\| \Big(\frac{1}{m_d(K, \rho_1)} \sum_{k=-K}^{K} \|f_2(k-1-m)\| \rho_2(k) \Big). \end{aligned}$$

Since $PAA_0S(\mathbb{Z}, X, \rho_1, \rho_2)$ is invariant under translation by [29, Lemma 10] we obtain that $f_2(\cdot - m) \in PAA_0S(\mathbb{Z}, X, \rho_1, \rho_2)$. By Lebesgue dominated convergence theorem, we have

$$\lim_{K \to \infty} \frac{1}{m_d(K, \rho_1)} \sum_{-K}^{K} \|u_2(k)\| \rho_2(k) = 0.$$

Hence $u \in WPAA_d(\mathbb{Z}, X)$. It proves the claim for such space. Now, let $f \in WPSAP_\omega(\mathbb{Z}, X, \rho)$. Then,

$$\begin{aligned} &\frac{1}{m_d(K, \rho)} \sum_{k=-K}^{K} \|u(k+\omega) - u(k)\| \rho(k) \\ &= \frac{1}{m_d(K, \rho)} \sum_{k=-K}^{K} \Big\| \sum_{j=-\infty}^{k+\omega-1} S_\alpha(k+\omega-1-j) f(j) \\ &- \sum_{j=-\infty}^{k-1} S_\alpha(k-1-j) f(j) \Big\| \rho(k) \\ &\leq \frac{1}{m_d(K, \rho)} \sum_{k=-K}^{K} \sum_{j=-\infty}^{k-1} \|S_\alpha(k-1-j)\| \|f(j+\omega) - f(j)\| \rho(k) \\ &\leq \sum_{m=0}^{\infty} \|S_\alpha(m)\| \Big(\frac{1}{m_d(K, \rho)} \sum_{k=-K}^{K} \|f(k-1-m+\omega) - f(k-1-m)\| \rho(k) \Big). \end{aligned}$$

Since $WPSAP_\omega(\mathbb{Z}, X, \rho)$ is invariant under translation by [30, Lemma 10] we obtain that $f(\cdot - 1 - m) \in WPSAP_\omega(\mathbb{Z}, X, \rho)$. By Lebesgue dominated convergence theorem, we have

$$\lim_{K \to \infty} \frac{1}{m_d(K, \rho)} \sum_{-K}^{K} \|u(k+\omega) - u(k)\| \rho(k) = 0.$$

Hence $u \in WPSAP_\omega(\mathbb{Z}, X, \rho)$. The proof is complete. $\square$

3. Solutions for nonlinear fractional difference equations

We consider the fractional difference equation

$$\Delta^\alpha u(n) = Au(n+1) + f(n, u(n)), \quad n \in \mathbb{Z}, \tag{3.1}$$

for $0 < \alpha < 1$, where A is the generator of a discrete α-resolvent family $\{S_\alpha(n)\}_{n\in\mathbb{N}_0}$ in $\mathcal{B}(X)$, and Δ^α is the fractional difference of the sequence u of order α.

Since our objective is to study the solubility of (3.1) in the spaces $\mathcal{M}(\mathbb{Z}, X)$, where the forcing term f is only bounded, we need to use the definition of mild solution introduced by Abadias and Lizama in [1].

Definition 3.1. Let $0 < \alpha < 1$, A be the generator of a discrete α-resolvent family $\{S_\alpha(n)\}_{n\in\mathbb{N}_0} \subset \mathcal{B}(X)$, and $f : \mathbb{Z} \times X \to X$. We say that a sequence $u : \mathbb{Z} \to X$ is a mild solution of (3.1) if $m \to S_\alpha(m)f(n-m)$ is summable on $\mathbb{N}_0$, for each $n \in \mathbb{Z}$ and u satisfies

$$u(n+1) = \sum_{j=-\infty}^{n} S_\alpha(n-j)f(j, u(j)), \quad n \in \mathbb{Z}.$$

Our first result in this section provides a simple criterion for the existence and uniqueness of discrete weighted pseudo almost automorphic and discrete $\mathcal{S}$-asymptotic ω-periodic mild solutions. The proof is based on the Banach fixed point theorem.

Theorem 3.2. *Let $0 < \alpha < 1$, $\rho_1, \rho_2 \in V_\infty$ and $\rho \in U_T$. Assume that A generates a summable discrete α-resolvent family $\{S_\alpha(n)\}_{n\in\mathbb{N}_0} \subset \mathcal{B}(X)$. If $f \in \Omega \in \mathcal{M}(\mathbb{Z} \times X, X)$ and it is globally Lipschitz in the following sense:*

$$\|f(n, x) - f(n, y)\| \le L\|x - y\|, \quad \text{for all } n \in \mathbb{Z} \text{ and all } x, y \in X,$$

where $L < \frac{1}{\|S_\alpha\|_1}$, then (3.1) has a unique mild solution u which belongs to the corresponding subset $\Omega \in \mathcal{M}(\mathbb{Z}, X)$.

Proof. Let $f \in WPAA_d(\mathbb{Z}\times X, X)$ and consider the operator $T : WPAA_d(\mathbb{Z}, X) \to WPAA_d(\mathbb{Z}, X)$ defined by

$$(Tu)(n) := \sum_{j=-\infty}^{n-1} S_\alpha(n-1-j)f(j, u(j)), \quad n \in \mathbb{Z}. \tag{3.2}$$

Since $u \in WPAA_d(\mathbb{Z}, X)$ it follows from Theorem 2.3 that $f(\cdot, u(\cdot))$ belongs to $WPAA_d(\mathbb{Z}, X)$. Now, from Theorem 2.13 we have that $Tu \in WPAA_d(\mathbb{Z}, X)$. Hence T is well-defined. In addition, for $u, v \in WPAA_d(\mathbb{Z}, X)$ and $n \in \mathbb{Z}$ the following inequality holds,

$$\begin{aligned}
\|(Tu)(n) - (Tv)(n)\| &\le \sum_{j=-\infty}^{n-1} \|S_\alpha(n-1-j)(f(j, u(j)) - f(j, v(j)))\| \\
&\le \sum_{j=-\infty}^{n-1} \|S_\alpha(n-1-j)\|\|u(j) - v(j)\| \\
&\le L\|S_\alpha\|_1\|u - v\|_\infty.
\end{aligned}$$

By hypothesis we conclude that T is a contraction, and using Banach fixed point theorem we get that there exists a unique discrete weighted pseudo almost automorphic mild solution of (3.1) .

The proof for the space of $\mathcal{S}$-asymptotic ω-periodic sequences is analogous. $\square$

Using [1, Remark 3.6] we have a precise estimate for $\|S_\alpha\|_1$ that can be used to prove the following Corollary. The proof use Theorem 2.12 and the proof of Theorem 3.2.

Corollary 3.3. *Let $0<\alpha<1$, $\rho_1,\rho_2 \in V_\infty$, $\rho\in U_T$ and A be the generator of a C_0-semigroup $T(t)$ such that $\|T(t)\|\le Me^{-\omega t}$, for some $M>0$ and $\omega>0$. If $f\in\mathcal{M}(\mathbb{Z}\times X,X)$ is globally Lipschitz with constant $L<1/\omega$ then (3.1) has a unique mild solution u which belongs to the same space as f.*

Next, we show that the conclusion of the previous theorem holds with a local Lipschitz condition on f.

Theorem 3.4. *Let $0<\alpha<1$, $\rho_1,\rho_2\in V_\infty$ and $\rho\in U_T$. Assume that A generates a summable discrete α-resolvent family $\{S_\alpha(n)\}_{n\in\mathbb{N}_0}\subset\mathcal{B}(X)$. Let $f\in\mathcal{M}(\mathbb{Z}\times X,X)$ that satisfies a local Lipschitz condition. If there exist $r_0>0$ such that*

$$\|S_\alpha\|_1\Big(L(r_0)+\frac{\sup_k\|f(k,0)\|}{r_0}\Big)<1,$$

then (3.1) has a unique mild solution u which belongs to the same space as f with $\|u\|_\infty:=\sup_k\|u(k)\|\le r_0$.

Proof. First, we consider $f\in WPAA_d(\mathbb{Z}\times X,X)$. Note that $T:WPAA_d(\mathbb{Z},X)\to WPAA_d(\mathbb{Z},X)$ given by (3.2) is well defined by Corollary 2.4 and Theorem 2.13.

Let $B_{r_0}(0):=\{u\in WPAA_d(\mathbb{Z},X):\|u\|_\infty<r_0\}$ be the ball of radius r_0 on $WPAA_d(\mathbb{Z},X)$. We show that $T\big(B_{r_0}(0)\big)\subset B_{r_0}(0)$. Indeed, let u be in $B_{r_0}(0)$. Since f is locally Lipschitz, we obtain

$$\|f(k,u(k))\|\le\|f(k,u(k))-f(k,0)\|+\|f(k,0)\|\le L(r_0)\|u(k)\|+\|f(k,0)\|,$$

for $k\in\mathbb{Z}$. Moreover, we have the estimate

$$\begin{aligned}
&\|T(u)(n)\|\\
&\le\sum_{j=-\infty}^{n-1}\|S_\alpha(n-1-j)\|\|f(j,u(j))-f(j,0)\|+\sum_{j=-\infty}^{n-1}\|S_\alpha(n-1-j)\|\|f(j,0)\|\\
&\le L(r_0)\sum_{j=-\infty}^{n-1}\|S_\alpha(n-1-j)\|\|u(j)\|+\|S_\alpha\|_1\sup_k\|f(k,0)\|\\
&\le\|S_\alpha\|_1\Big(L(r_0)+\frac{\sup_k\|f(k,0)\|}{r_0}\Big)r_0\le r_0,
\end{aligned}$$

proving the claim. On the other hand, for $u,v\in B_{r_0}(0)$ we have that

$$\begin{aligned}
&\|Tu(n)-Tv(n)\|\\
&\le\sum_{j=-\infty}^{n-1}\|S_\alpha(n-1-j)\|\|f(j,u(j))-f(j,v(j))\|\\
&\le L(r_0)\sum_{j=-\infty}^{n-1}\|S_\alpha(n-1-j)\|\|u(j)-v(j)\|\\
&\le\|S_\alpha\|_1L(r_0)\|u-v\|_\infty.
\end{aligned}$$

Observing that $\|S_\alpha\|_1 L(r_0) < 1$, it follows that T is a contraction in $B_{r_0}(0)$. Then there is a unique $u \in B_{r_0}(0)$ such that $Tu = u$.

The proof for $f \in WPSAP_\omega(\mathbb{Z} \times X, X, \rho)$ is similar, we just have to take $B_{r_0}(0) := \{u \in WPSAP_\omega(\mathbb{Z}, X, \rho) : \|u\|_\infty < r_0\}$ be the ball of radius r_0 on $WPSAP_\omega(\mathbb{Z}, X, \rho)$. $\square$

The following corollary is an immediate consequence of the previous results.

Corollary 3.5. *Let $0 < \alpha < 1$, $\rho_1, \rho_2 \in U_\infty$, $\rho \in U_T$ and A be the generator of a C_0-semigroup $T(t)$ such that $\|T(t)\| \le Me^{-\omega t}$, for some $M > 0$ and $\omega > 0$. If $f \in \mathcal{M}(\mathbb{Z} \times X, X)$ is locally Lipschitz and satisfy*

$$\frac{1}{\omega}\Big(L(r_0) + \frac{\sup_k \|f(k,0)\|}{r_0}\Big) < 1,$$

for some $r_0 > 0$, then (3.1) *has a unique mild solution u which belongs to the same space as f.*

We finish this article with a simple example to illustrate how our abstract results apply.

Example 3.6. We consider the fractional difference equation

$$\Delta^\alpha u(k) = Au(k+1) + \frac{\nu g(k) u(k)}{1 + \sup_k |u(k)|}, \quad k \in \mathbb{Z}, \tag{3.3}$$

where $0 < \alpha < 1$ act as a tuning parameter for the difference Equation (3.3), the operator A is the generator of an exponentially stable C_0-semigroup on a Banach space X, ν is a parameter and $g(k) = \mathrm{signum}(\cos 2\pi k\theta) + e^{-|k|}$. We know by [2] that $g \in WPAA_d(\mathbb{Z}, X)$ where $\rho = 1 + k^2$. Now, it can be shown that the function

$$f(k,x) := \frac{\nu g(k) x}{1 + \|x\|_\infty}, \quad k \in \mathbb{Z},\, x \in X,$$

is a discrete weighted pseudo almost automorphic function on $\mathbb{Z} \times X$. We have the estimate

$$\|f(k,x) - f(k,y)\|_\infty \le \nu \|g\|_\infty (1 + \|y\|)\|x - y\|_\infty.$$

Therefore, we can choose $L(r) = \nu\|g\|_\infty(1 + r)$, $r > 0$, to deduce that $f(k,x)$ is locally Lipschitz. Since $f(k,0) = 0$, we obtain that for sufficiently small ν the condition $\|S_\alpha\|_1 L(r) < 1$ is satisfied. We conclude, by Theorem 3.4, that the fractional model (3.3) admits a unique discrete weighted pseudo almost automorphic solution.

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