# WEIGHTED PSEUDO ANTIPERIODIC SOLUTIONS FOR FRACTIONAL INTEGRO-DIFFERENTIAL EQUATIONS IN BANACH SPACES.

EDGARDO ALVAREZ, CARLOS LIZAMA*†, AND RODRIGO PONCE

ABSTRACT. In this paper we prove the existence of weighted pseudo antiperiodic mild solutions for fractional integro-differential equations in the form

$$D^{\alpha}(t) = Au(t) + \int_{-\infty}^{t} a(t-s)Au(s)\,ds + f(t,u(t)),$$

where $f(\cdot,u(\cdot))$ is Stepanov weighted pseudo antiperiodic and $A$ generates a resolvent family of bounded and linear operators on a Banach space $X$, $a \in L^1_{\mathrm{loc}}(\mathbb{R}_+)$ and $\alpha > 0$. Also, we give a short proof to show that the vector-valued space of Stepanov-like weighted pseudo antiperiodic functions is a Banach space.

## 1. INTRODUCTION

Let us consider the equation

$$L(u) = f, \tag{1.1}$$

where $L$ is a linear, possibly unbounded operator, and the forcing term $f$ belongs to some space of vector-valued functions, say $\mathcal{M}$. It is well known that mathematical understanding of the linear Equation (1.1) is meant as a preliminary critical step for the subsequent analysis of full nonlinear models. Usually, one is interested in to find conditions on the operator $L$ such that the solution $u$ belongs to the same space of vector-valued functions than $f$. Then, fixed point arguments are used to obtain the desired solution of associated nonlinear problems.

We ask ourselves the following: $(Q)$ Can the solution $u$ be more regular that $f$?. In other words, is it possible to find a subspace $\mathcal{N} \subset \mathcal{M}$ such that $u \in \mathcal{N}$?.

This problem has begun to be studied in the last years and there are some cases in the literature where the answer is positive. For instance, Diagana, N'Guérékata and Mophou [8] solved the problem $(Q)$ taking $\mathcal{M}$ as the space of Stepanov-like weighted pseudo almost automorphic functions, $L(u) := D^{\alpha}_t u - Au$ and $\mathcal{N}$ as the subspace of weighted pseudo almost automorphic functions. Here $A$ is a closed and linear operator defined on a Banach space $X$ and $D^{\alpha}$ denotes fractional derivative of order $\alpha > 0$. A further generalization when the forcing term $f$ also depends on a given operator was studied recenty by Mishra

---

2010 *Mathematics Subject Classification.* 35B15; 47D06; 45D05.

*Key words and phrases.* antiperiodic; weighted; pseudo; Stepanov; mild solutions; Banach space; fractional; integro-differential.

**Corresponding author.* Universidad de Santiago de Chile, Departamento de Matemática y Ciencia de la Computación, Las Sophoras 173, Estación Central, Santiago, Chile. Tel.: 56-2-7182035.

† Partially supported by Proyecto Anillo ACT 1112 and FONDECYT 1140258.

and Bahuguna [12]. Previously, Diagana [6] studied the case $\alpha = 1$ with $\mathcal{M}$ the space of Stepanov-like pseudo almost automorphic functions and $\mathcal{N}$ the subspace of pseudo almost automorphic functions. The nonautonomous case, that is $L(u) = u' - A(t)u$, was very recently studied by Xia [17, Theorem 42]. He consider $\mathcal{M}$ as the space of Stepanov-like weighted pseudo periodic functions and $\mathcal{N}$ as the subspace of weighted pseudo periodic functions. We also note that Xia and Fan [18] have solved problem $(Q)$ for some interesting classes of partial neutral functional differential equations, nonlinear Volterra equations of scalar type and semilinear equations with delay.

In this paper, we are able to give an affirmative answer to (Q) taking $\mathcal{M}$ as the space of Stepanov-like weighted pseudo antiperiodic functions; $\mathcal{N}$ as the space of weighted pseudo antiperiodic functions and the class of operators is defined by

$$L(u)(t) = D^{\alpha}u(t) - Au(t) - \int_{-\infty}^{t} a(t-s)Au(s)\,ds, \tag{1.2}$$

where $A$ generates a resolvent family $\{S_{\alpha}(t)\}_{t\geq 0}$ on Banach space $X$, $a \in L^1_{\text{loc}}(\mathbb{R}_+)$ and $\alpha > 0$. Equation (1.1) with $L$ defined by (1.2) has been in studied in [15]. The study of existence of solutions to such class of fractional differential equations is an important topic due to its significance and applications in physics, probability, modelling, mechanics and other areas. However, to the best of our knowledge, the existence of weighted pseudo antiperiodic solutions to (1.2) in the case when the forcing term $f$ is Stepanov weighted pseudo antiperiodic is an untreated original problem, which constitutes one of the main motivations of this work.

Parallell to this, we note that the existence and uniqueness of antiperiodic solutions to evolution equations is an important topic that have been studied in several works. We mention here Aftabizadeh, Aizicivici and Pavel [1], [2], Al-Islam, Alsulami and Diagana [3], H.L.Chen [4], Y.Q. Chen [5], Haraux [9], Okoshi [14], and N'Guérékata and Valmorin [13].

This paper is organized as follows. In Section 2, we first present some definitions and basic results of Stepanov-like type spaces and then we give a short and direct proof to the fact that the space of Stepanov-like weighted pseudo antiperiodic functions is a Banach space (Theorem 2.15). In Section 3, we first prove a composition Theorem in the space of Stepanov-like weighted pseudo antiperiodic functions, assuming a compactness condition (Theorem 3.3). Then, we show sufficient conditions in order to ensure the existence and uniqueness of weighted weighted antiperiodic mild solutions where the input data $f$ belongs to the space of Stepanov-like weighted pseudo antiperiodic functions. We finish this paper with an illustrative example to find existence and uniqueness of mild solutions for a concrete semilinear problem is given.

## 2. Preliminaries

In this section, we introduce some basic definitions, notations and preliminaries facts that we will use in the paper. Particularly, we give an alternative proof to show that the space of Stepanov-like weighted pseudo antiperiodic functions is a Banach space.

Throughout the paper $(X, \|\cdot\|_X)$ and $(Y, \|\cdot\|_Y)$ are complex Banach spaces and $B(X,Y)$ is the Banach space of bounded linear operators from $X$ to $Y$; when $X = Y$ we simply write $B(X)$.

We denote by

$$BC(\mathbb{R}, X) := \{f : \mathbb{R} \to X : f \text{ is continuous, } ||f||_\infty := \sup_{t\in\mathbb{R}} ||f(t)|| < \infty\},$$

the Banach space of $X$-valued bounded and continuous functions on $\mathbb{R}$, with natural norm.

Given a function $g : \mathbb{R} \to X$, the *Caputo (or Weyl) fractional integral* of order $\alpha > 0$ is defined by

$$D^{-\alpha}g(t) := \frac{1}{\Gamma(\alpha)} \int_{-\infty}^{t} (t-s)^{\alpha-1} g(s)ds, \quad t \in \mathbb{R},$$

when this integral is convergent. The *Caputo (or Weyl) fractional derivative* $D^\alpha g$ of order $\alpha > 0$ is defined by

$$D^\alpha g(t) := \frac{d^n}{dt^n} D^{-(n-\alpha)} g(t), \quad t \in \mathbb{R},$$

where $n = [\alpha] + 1$. It is known that $D^\alpha D^{-\alpha} g = g$ for any $\alpha > 0$, and $D^n = \frac{d^n}{dt^n}$ holds with $n \in \mathbb{N}$. See [11] for more details.

The Mittag-Leffler function (see e.g. [10]) is defined as follows:

$$E_{\alpha,\beta}(z) := \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(\alpha k + \beta)} = \frac{1}{2\pi i} \int_{Ha} e^{\mu} \frac{\mu^{\alpha-\beta}}{\mu^\alpha - z} d\mu, \quad \alpha, \beta > 0, z \in \mathbb{C},$$

where $Ha$ is a Hankel path, i.e. a contour which starts and ends at $-\infty$ and encircles the disc $|\mu| \le |z|^{1/\alpha}$ counterclockwise. The Laplace transform of a variant of the Mittag-Leffler function is given by:

$$\mathcal{L}(t^{\beta-1} E_{\alpha,\beta}(-\rho t^\alpha))(\lambda) = \frac{\lambda^{\alpha-\beta}}{\lambda^\alpha + \rho}, \quad \rho \in \mathbb{C}, \mathrm{Re}\lambda > |\rho|^{1/\alpha}.$$

We recall the following definition [15] (see also [16] for a general treatment on resolvent families).

**Definition 2.1.** *Let $A$ be a closed and linear operator with domain $D(A)$ defined on a Banach space $X$, and $\alpha > 0$. Given $a \in L^1_{\mathrm{loc}}(\mathbb{R}_+)$, we say that $A$ is the generator of an $\alpha$-resolvent family, if there exist $\omega \ge 0$ and a strongly continuous function $S_\alpha : [0,\infty) \to \mathcal{B}(X)$ such that $\{\frac{\lambda^\alpha}{1+\hat{a}(\lambda)} : \mathrm{Re}\lambda > \omega\} \subset \rho(A)$, the resolvent set of $A$, and for all $x \in X$,*

$$\big(\lambda^\alpha - (1+\hat{a}(\lambda))A\big)^{-1} x = \frac{1}{1+\hat{a}(\lambda)} \left( \frac{\lambda^\alpha}{1+\hat{a}(\lambda)} - A \right)^{-1} x = \int_0^\infty e^{-\lambda t} S_\alpha(t) x dt, \ \mathrm{Re}\lambda > \omega.$$

*In this case, $\{S_\alpha(t)\}_{t\ge0}$ is called the $\alpha$-*resolvent family *generated by $A$.*

Now, we recall the definitions of antiperiodic functions.

**Definition 2.2.** *A function $f \in C(\mathbb{R}, X)$ is said to be antiperiodic if there exists a $\omega \in \mathbb{R} \setminus \{0\}$ with the property $f(t+\omega) = -f(t)$ for all $t \in \mathbb{R}$. If there exists a least positive $\omega$ with this property, it is called the anti-period of $f$. The collection of those functions with the same anti-period $\omega$ is denoted by $P_{\omega ap}(\mathbb{R}, X)$.*

**Remark 2.3.** *Note that if* $f \in P_{\omega ap}(\mathbb{R}, X)$*, then* $f \in P_{2\omega}(\mathbb{R}, X)$*, where* $P_{2\omega}(\mathbb{R}, X)$ *denotes the Banach space of all* $2\omega$*-periodic functions.*

**Definition 2.4.** *A function* $f \in C(\mathbb{R} \times X, X)$ *(resp.,* $C(\mathbb{R} \times X \times X, X)$*) is said to be antiperiodic in* $t \in \mathbb{R}$ *and uniformly in* $u \in X$ *(resp. in* $(u, v) \in X \times X$*) if there exists a* $\omega \in \mathbb{R} \setminus \{0\}$ *with the property* $f(t + \omega, u) = -f(t, u)$ *for all* $t \in \mathbb{R}$, $u \in X$*. (resp.* $f(t + \omega, u, v) = -f(t, u, v)$ *for all* $t \in \mathbb{R}$, $(u, v) \in X \times X$*). The collection of those* $\omega$*-antiperiodic functions is denoted by* $P_{\omega ap}(\mathbb{R} \times X, X)$ *(resp.,* $P_{\omega ap}(\mathbb{R} \times X \times X, X)$*).*

Let $U$ be denote the set of all functions $\rho : \mathbb{R} \to (0, \infty)$ in $L^1_{loc}(\mathbb{R})$ such that $\rho(t) > 0$ for all $t \in \mathbb{R}$ a.e. For a given $r > 0$ and for each $\rho \in U$, we set

$$m(r, \rho) := \int_{-r}^{r} \rho(t)\, dt.$$

Thus the space of weights $U_\infty$ is defined by

$$U_\infty := \{\rho \in U : \lim_{r\to\infty} m(r, \rho) = \infty\}.$$

Now, for $\rho \in U_\infty$, we define

$$PAA_0(\mathbb{R}, X) := \{f \in BC(\mathbb{R}, X) : \lim_{r\to\infty} \frac{1}{m(r, \rho)} \int_{-r}^{r} \|f(t)\|\rho(t)\, dt = 0\};$$

$$\begin{aligned} PAA_0(\mathbb{R} \times Y, X) := \{f \in BC(\mathbb{R} \times Y, X) \quad &: \quad f(\cdot, y) \text{ is bounded for each } y \in Y \\ \text{and} \lim_{r\to\infty} \frac{1}{m(r, \rho)} \int_{-r}^{r} \|f(t, y)\|\rho(t)\, dt \quad &= \quad 0, \quad \text{uniformly in } y \in Y\}. \end{aligned}$$

**Definition 2.5** ([7])**.** *Let* $\rho \in U_\infty$*. A function* $f \in BC(\mathbb{R}, X)$ *(respectively* $f \in BC(\mathbb{R} \times Y, X)$*) is called weighted pseudo antiperiodic if it can be expressed as* $f = g + h$ *where* $g \in P_{\omega ap}(\mathbb{R}, X)$ *(respectively* $P_{\omega ap}(\mathbb{R} \times Y, X)$*) and* $h \in PAA_0(\mathbb{R}, X)$ *(respectively* $PAA_0(\mathbb{R} \times Y, X)$*). We denote by* $WPP_{\omega ap}(\mathbb{R}, X)$ *(respectively* $WPP_{\omega ap}(\mathbb{R} \times Y, X)$*) the set of all such functions.*

**Definition 2.6** ([8])**.** *The Bochner transform* $f^b(t, s)$ *with* $t \in \mathbb{R}, s \in [0, 1]$ *of a function* $f : \mathbb{R} \to X$ *is defined by*

$$f^b(t, s) := f(t + s).$$

**Definition 2.7** ([8])**.** *The Bochner transform* $f^b(t, s, u)$ *with* $t \in \mathbb{R}, s \in [0, 1], u \in X$ *of a function* $f : \mathbb{R} \times X \to X$ *is defined by*

$$f^b(t, s, u) := f(t + s, u) \quad \text{for all} \quad u \in X.$$

**Definition 2.8** ([8])**.** *Let* $p \in [1, \infty)$*. The space* $BS^p(\mathbb{R}, X)$ *of all Stepanov bounded functions, with exponent* $p$*, consist of all measurable functions* $f : \mathbb{R} \to X$ *such that* $f^b \in L^\infty(\mathbb{R}, L^p(0, 1; X))$*. This is a Banach space with the norm*

$$\|f\|_{S^p} := \|f^b\|_{L^\infty(\mathbb{R}, L^p)} = \sup_{t\in\mathbb{R}} \left( \int_t^{t+1} \|f(\tau)\|^p\, d\tau \right)^{\frac{1}{p}}.$$

**Definition 2.9.** *A function* $f \in BS^p(\mathbb{R}, X)$ *is called Stepanov antiperiodic if* $f^b \in P_{\omega ap}(\mathbb{R}, L^p(0, 1; X))$*. We denote the set of all such functions by* $P_{\omega ap}S^p(\mathbb{R}, X)$*.*

**Remark 2.10.** *We note that the preceding definition implies*

$$\sup_{t\in\mathbb{R}}\left(\int_t^{t+1}\|f(s+\omega)+f(s)\|^p\,ds\right)^{1/p}=0$$

*which is equivalent to say that* $f(t+\omega)=-f(t)$ *a.e.* $t\in\mathbb{R}$*; that is;* $\|f(t+\omega)+f(t)\|_p=0$*. We observe that this coincide with the definition of Xia in* [17].

**Definition 2.11.** *A function* $f:\mathbb{R}\times X\to Y$ *with* $f(\cdot,u)\in BS^p(\mathbb{R},Y)$*, for each* $u\in X$*, is called Stepanov antiperiodic function in* $t\in\mathbb{R}$ *uniformly for* $u\in X$ *if* $f(t+\omega,u)=-f(t,u)$ *a.e.* $t\in\mathbb{R}$ *and each* $u\in X$*. We denote by* $P_{\omega ap}S^p(\mathbb{R}\times X,Y)$ *the set of all such functions.*

Now, we introduce a (natural) linear operator from $BS^p(\mathbb{R},X)$ into $L^\infty(\mathbb{R},L^p(0,1;X))$ which will be an important tool in order to clarity some concepts and achieve our goals.

**Definition 2.12.** *We define the map*

$$\begin{aligned}\mathcal{B}:&BS^p(\mathbb{R},X)\to L^\infty(\mathbb{R},L^p(0,1;X))\\ &f\mapsto(\mathcal{B}f)(t)(s)=f(t+s).\end{aligned}$$

**Remark 2.13.** *It follows from the definitions that the operator* $\mathcal{B}$ *is a linear isometry between* $BS^p(\mathbb{R},X)$ *and* $L^\infty(\mathbb{R},L^p(0,1;X))$*. More precisely*

$$\|\mathcal{B}f\|_\infty=\|f\|_{BS^p(\mathbb{R},X)}.$$

**Remark 2.14.** *The definition of Stepanov-like weighted pseudo antiperiodic functions given by Xia in* [17] *can be rewritten using the preceding notation. Thus, for* $\rho\in U_\infty$*, we say that a function* $f$ *is Stepanov-like weighted pseudo antiperiodic (or* $S^p-$*weighted pseudo antiperiodic) if and only if* $f\in\mathcal{B}^{-1}(P_{\omega ap}(\mathbb{R},L^p(0,1;X)))+\mathcal{B}^{-1}(PAA_0(\mathbb{R},L^p(0,1;X)))$*, where by notation* $\mathcal{B}^{-1}(M):=\{f\in BS^p(\mathbb{R},X):\mathcal{B}(f)=g \text{ for some } g\in M\}$ *for* $M\subseteq L^\infty(\mathbb{R},L^p(0,1;X))$ *In other words,*

$$WPP_{\omega ap}S^p(\mathbb{R},X)=\mathcal{B}^{-1}(P_{\omega ap}(\mathbb{R},L^p(0,1;X)))+\mathcal{B}^{-1}(PAA_0(\mathbb{R},L^p(0,1;X)))\tag{2.1}$$

*Moreover, since* $\mathcal{B}$ *is an isometry and* $P_{\omega ap}(\mathbb{R},L^p(0,1;X))\cap PAA_0(\mathbb{R},L^p(0,1;X))=\{0\}$ *then the sum in* (2.1) *is direct, that is,*

$$WPP_{\omega ap}S^p(\mathbb{R},X)=\mathcal{B}^{-1}(P_{\omega ap}(\mathbb{R},L^p(0,1;X)))\oplus\mathcal{B}^{-1}(PAA_0(\mathbb{R},L^p(0,1;X))).$$

Based in the definition of operator $\mathcal{B}$, we prove that $WPP_{\omega ap}S^p(\mathbb{R},X)$ is a Banach space when endowed with their natural norm.

**Theorem 2.15.** $WPP_{\omega ap}S^p(\mathbb{R},X)$ *is a Banach space with the norm*

$$\|f\|_{WPP_{\omega ap}S^p(\mathbb{R},X)}:=\|g\|_{BS^p(\mathbb{R},X)}+\|h\|_{BS^p(\mathbb{R},X)}$$

*where* $f=g+h$ *with* $g\in\mathcal{B}^{-1}(P_{\omega ap}(\mathbb{R},L^p(0,1;X)))$ *and* $h\in\mathcal{B}^{-1}(PAA_0(\mathbb{R},L^p(0,1;X)))$*.*

*Proof.* Let $(f_n)$ be a Cauchy sequence in $WPP_{\omega ap}S^p(\mathbb{R},X)$. Then $\|f_n-f_m\|_{WPP_{\omega ap}S^p(\mathbb{R},X)}\to 0$ as $n,m\to\infty$. Let $f_n=g_n+h_n$ and $f_m=g_m+h_m$ with $g_n,g_m\in\mathcal{B}^{-1}(P_{\omega ap}(\mathbb{R},L^p(0,1;X)))$ and $h_n,h_m\in\mathcal{B}^{-1}(PAA_0(\mathbb{R},L^p(0,1;X)))$. If $n,m\to\infty$, then

$$\|\mathcal{B}g_n-\mathcal{B}g_m\|_{L^\infty(\mathbb{R},L^p)}=\|g_n-g_m\|_{BS^p(\mathbb{R},X)}\leq\|f_n-f_m\|_{WPP_{\omega ap}S^p(\mathbb{R},X)}\to 0$$

and

$$\|\mathcal{B}g_n - \mathcal{B}g_m\|_{L^\infty(\mathbb{R},L^p)} = \|g_n - g_m\|_{BS^p(\mathbb{R},X)} \le \|f_n - f_m\|_{WPP_{\omega ap}S^p(\mathbb{R},X)} \to 0.$$

This implies that $(\mathcal{B}g_n)$ and $(\mathcal{B}h_n)$ are Cauchy sequences in $P_{\omega ap}(\mathbb{R}, L^p(0,1;X))$ and $PAA_0(\mathbb{R}, L^p(0,1;X))$ respectively. Since $P_{\omega ap}(\mathbb{R}, L^p(0,1;X))$ and $PAA_0(\mathbb{R}, L^p(0,1;X))$ are Banach spaces (see [13] and [**?**] resp.) then there exist $g \in P_{\omega ap}(\mathbb{R}, L^p(0,1;X))$ and $h \in PAA_0(\mathbb{R}, L^p(0,1;X))$ such that

$$\|\mathcal{B}g_n - g\|_{L^\infty(\mathbb{R},L^p)} \to 0, \quad \|\mathcal{B}h_n - h\|_{L^\infty(\mathbb{R},L^p)} \to 0 \quad (n \to \infty).$$

Let $f_1 := \mathcal{B}^{-1}(\{g\}) \in \mathcal{B}^{-1}(P_{\omega ap}(\mathbb{R}, L^p(0,1;X)))$ and $f_2 := \mathcal{B}^{-1}(\{h\}) \in \mathcal{B}^{-1}(PAA_0(\mathbb{R}, L^p(0,1;X)))$. Note that $f_1$ and $f_2$ are uniquely defined because $\mathcal{B}$ is injective. Let $f := f_1 + f_2 \in WPP_{\omega ap}S^p(\mathbb{R}, X)$. Then

$$\begin{aligned}
\|f_n - f\|_{WPP_{\omega ap}S^p(\mathbb{R},X)} &= \|(g_n + h_n) - (f_1 + f_2)\|_{WPP_{\omega ap}S^p(\mathbb{R},X)} \\
&= \|g_n - f_1\|_{BS^p(\mathbb{R},X)} + \|h_n - f_2\|_{BS^p(\mathbb{R},X)} \\
&= \|\mathcal{B}g_n - \mathcal{B}f_1\|_{L^\infty(\mathbb{R},L^p)} + \|\mathcal{B}h_n - \mathcal{B}f_2\|_{L^\infty(\mathbb{R},L^p)} \\
&= \|\mathcal{B}g_n - g\|_{L^\infty(\mathbb{R},L^p)} + \|\mathcal{B}h_n - h\|_{L^\infty(\mathbb{R},L^p)} \to 0 \quad (n \to \infty).
\end{aligned}$$

Therefore $WPP_{\omega ap}S^p(\mathbb{R}, X)$ is a Banach space. □

**Theorem 2.16.** *Let $\rho \in U_\infty$ be given and let $S : \mathbb{R}_+ \to B(X)$ be strongly continuous. Suppose that there exist a function $\phi \in L^1(\mathbb{R}_+)$ such that*

*(a) $\|S(t)\| \le \phi(t) \quad t \ge 0$;*
*(b) $\phi(t)$ is increasing;*
*(c) $\sum_{n=0}^{\infty} \phi(n) < \infty$.*

*Suppose that $f \in WPP_{\omega ap}S^p(\mathbb{R}, X)$. Then*

$$(S * f)(t) := \int_{-\infty}^{t} S(t-s)f(s)\,ds \in WPP_{\omega ap}(\mathbb{R}, X).$$

*Proof.* See [17, Lemma 36]. □

## 3. WEIGHTED PSEUDO ANTIPERIODIC MILD SOLUTIONS

In this section we consider the problem of existence and uniqueness of weighted pseudo antiperiodic mild solutions for the following class of integro-differential equations

$$D^\alpha u(t) = Au(t) + \int_{-\infty}^{t} a(t-s)Au(s)\,ds + f(t, u(t)), \tag{3.1}$$

where $A$ generates an $\alpha$-resolvent family $\{S_\alpha(t)\}_{t\ge0}$ on a Banach space $X$, $a \in L^1_{\mathrm{loc}}(\mathbb{R}_+)$, $\alpha > 0$ and the fractional derivative is understood in the sense of Caputo. Note that Equation (3.1) has the form of Equation (1.1) with $Lu = D^\alpha u(t) - Au(t) - \int_{-\infty}^{t} a(t-s)Au(s)\,ds$.

**Definition 3.1.** *A function* $u : \mathbb{R} \to X$ *is said to be a mild solution of* (3.1) *if*

$$u(t) = \int_{-\infty}^{t} S_\alpha(t-s) f(s, u(s))\, ds \quad (t \in \mathbb{R})$$

*where* $\{S_\alpha(t)\}_{t \geq 0}$ *is the* $\alpha$*-resolvent family generated by* $A$*, whenever it exists.*

Now, we present the following composition theorems.

**Theorem 3.2.** *Assume that* $F : \mathbb{R} \times X \to X$ *is a bounded function that satisfies*

- *(a) There exists* $\omega > 0$ *such that* $F(t+\omega, -x) = -F(t,x)$ *for a.e.* $t \in \mathbb{R}$ *and for all* $x \in X$*;*
- *(b) There exists* $L > 0$ *such that* $\|F(t,x) - F(t,y)\| \leq L\|x-y\|$ *for all* $x, y \in X$ *and* $t \in \mathbb{R}$*;*
- *(c)* $u \in P_{\omega ap}S^p(\mathbb{R}, X)$*.*

*Then* $F(\cdot, u(\cdot)) \in P_{\omega ap}S^p(\mathbb{R}, X)$*.*

*Proof.* Since $\|F(t,x) - F(t,y)\| \leq L\|x-y\|$ implies $\|F(t,x) - F(t,y)\|_p \leq L\|x-y\|_p$, then

$$\begin{aligned} \|F(t+\omega, u(t+\omega)) + F(t, u(t))\|_p &= \|F(t+\omega, u(t+\omega)) - F(t+\omega, -u(t))\|_p \\ &\quad + \|F(t+\omega, -u(t)) + F(t, u(t))\|_p \\ &\leq L\|u(t+\omega) + u(t)\|_p \\ &\quad + \|F(t+\omega, -u(t)) + F(t, u(t))\|_p = 0. \end{aligned}$$

Therefore $F(t+\omega, u(t+\omega)) = -F(t, u(t))$ a.e. $t \in \mathbb{R}$ and consequently $F(\cdot, u(\cdot)) \in P_{\omega ap}(\mathbb{R}, X)$. ☐

Our next result assume a compactness condition in order to obtain invariance under composition of functions for the space of Stepanov weighted pseudo antiperiodic functions.

**Theorem 3.3.** *Let* $\rho \in U_\infty$*,* $p > 1$*,* $f = g + \phi \in WPP_{\omega ap}S^p(\mathbb{R} \times X, X)$ *with* $g \in \mathcal{B}^{-1}(P_{\omega ap}(\mathbb{R} \times X, L^p(0,1;X)))$ *and* $\phi \in \mathcal{B}^{-1}(PAA_0(\mathbb{R} \times X, L^p(0,1;X)))$*. Assume that*

- *(i) There exists* $\omega > 0$ *such that* $f(t+\omega, -x) = -f(t,x)$*.*
- *(ii) There exist constants* $L_f, L_g > 0$ *such that*

  $$\|f(t,u) - f(t,v)\| \leq L_f\|u-v\|, \;\; \|g(t,u) - g(t,v)\| \leq L_g\|u-v\| \quad t \in \mathbb{R},\, u, v \in X.$$
- *(iii)* $h = \alpha + \beta \in WPP_{\omega ap}S^p(\mathbb{R}, X)$ *with* $\alpha \in \mathcal{B}^{-1}(P_{\omega ap}(\mathbb{R}, L^p(0,1;X)))$ *and* $\beta \in \mathcal{B}^{-1}(PAA_0(\mathbb{R}, L^p(0,1;X)))$ *is such that the set*

  $$K := \overline{\{\alpha(t) : t \in \mathbb{R}\}}$$

  *is compact in* $X$*. Then* $f(\cdot, h(\cdot)) \in WPP_{\omega ap}S^p(\mathbb{R}, X)$*.*

*Proof.* We can decompose

$$f(t, h(t)) = g(t, \alpha(t)) + f(t, h(t)) - f(t, \alpha(t)) + \phi(t, \alpha(t)).$$

Set

$$F(t) := g(t, \alpha(t)), \;\; G(t) := f(t, h(t)) - f(t, \alpha(t)), \;\; H(t) := \phi(t, \alpha(t)).$$

Since $\alpha \in P_{\omega ap}S^p(\mathbb{R}, X)$ and $g \in P_{\omega ap}S^p(\mathbb{R} \times X, X)$ then by assumptions and Theorem 3.2 we obtain that $F(t) \in \mathcal{B}^{-1}(P_{\omega ap}(\mathbb{R}, L^p(0,1;X)))$.

Next we show that $G(t) \in \mathcal{B}^{-1}(PAA_0(\mathbb{R}, L^p(0,1;X)))$. Indeed

$$\begin{aligned}\int_t^{t+1} \|G(\sigma)\|^p \, d\sigma &= \int_t^{t+1} \|f(\sigma, h(\sigma)) - f(\sigma, \alpha(\sigma))\|^p \, d\sigma \\ &\leq \int_t^{t+1} L_f^p \|h(\sigma) - \alpha(\sigma)\|^p \, d\sigma \\ &= \int_t^{t+1} L_f^p \|\beta(\sigma)\|^p \, d\sigma.\end{aligned}$$

Then

$$\frac{1}{m(r,\rho)} \int_{-r}^{r} \left( \int_t^{t+1} \rho(t) \|G(\sigma)\|^p \, d\sigma \right)^{1/p} dt \leq \frac{L_f}{m(r,\rho)} \int_{-r}^{r} \rho(t) \left( \int_t^{t+1} \|\beta(\sigma)\|^p \, d\sigma \right)^{1/p} dt.$$

Since $\beta(\cdot) \in \mathcal{B}^{-1}(PAA_0(\mathbb{R}, L^p(0,1;X)))$ we obtain that $G(\cdot) \in \mathcal{B}^{-1}(PAA_0(\mathbb{R}, L^p(0,1;X)))$.

Next, we prove that $H(\cdot) \in \mathcal{B}^{-1}(PAA_0(\mathbb{R}, L^p(0,1;X)))$. Since $\phi \in \mathcal{B}^{-1}(PAA_0(\mathbb{R} \times X, L^p(0,1;X)))$ then for any $\epsilon > 0$ there exist $r_0 > 0$ such that $r > r_0$ implies that

$$\frac{1}{m(r,\rho)} \int_{-r}^{r} \rho(t) \left( \int_t^{t+1} \|\phi(\sigma, u)\|^p \, d\sigma \right)^{1/p} dt < \epsilon \quad (u \in X).$$

Since $K$ is compact, we can find finite open balls $O_k$ $(k = 1, 2, 3, ..., n)$ with center $x_k$ and radius less than $\frac{\epsilon}{L_f + L_g}$ such that $K \subset \cup_{k=1}^{m} O_k$. Set $B_k := \{t \in \mathbb{R} : \alpha(t) \in O_k\}$. Then $\mathbb{R} = \cup_{k=1}^{n} B_k$. Let $E_1 = B_1$, $E_k = B_k \setminus (\cup_{j=1}^{k-1} B_j)$ $(2 \leq k \leq m)$. Thus $E_i \cap E_j = \emptyset$ for $i \neq j$.

By Minkowski inequality, for $r > r_0$ we have

$$\begin{aligned}
&\frac{1}{m(r,\rho)} \int_{-r}^{r} \rho(t) \left( \int_t^{t+1} \|\phi(\sigma, \alpha(\sigma))\|^p \, d\sigma \right)^{1/p} dt \\
&= \frac{1}{m(r,\rho)} \sum_{k=1}^{n} \int_{E_k \cap [-r,r]} \rho(t) \left( \int_t^{t+1} \|\phi(\sigma, \alpha(\sigma))\|^p \, d\sigma \right)^{1/p} dt \\
&\le \frac{1}{m(r,\rho)} \sum_{k=1}^{n} \int_{E_k \cap [-r,r]} \rho(t) \left( \int_t^{t+1} \|\phi(\sigma, \alpha(\sigma)) - \phi(\sigma, \alpha(x_k))\|^p \, d\sigma \right)^{1/p} dt \\
&+ \frac{1}{m(r,\rho)} \sum_{k=1}^{n} \int_{E_k \cap [-r,r]} \rho(t) \left( \int_t^{t+1} \|\phi(\sigma, \alpha(x_k))\|^p \, d\sigma \right)^{1/p} dt \\
&\le \frac{1}{m(r,\rho)} \sum_{k=1}^{n} \int_{E_k \cap [-r,r]} \rho(t) \left( \int_t^{t+1} \|f(\sigma, \alpha(\sigma)) - f(\sigma, \alpha(x_k))\|^p \, d\sigma \right)^{1/p} dt \\
&+ \frac{1}{m(r,\rho)} \sum_{k=1}^{n} \int_{E_k \cap [-r,r]} \rho(t) \left( \int_t^{t+1} \|g(\sigma, \alpha(\sigma)) - g(\sigma, \alpha(x_k))\|^p \, d\sigma \right)^{1/p} dt \\
&+ \frac{1}{m(r,\rho)} \sum_{k=1}^{n} \int_{E_k \cap [-r,r]} \rho(t) \left( \int_t^{t+1} \|\phi(\sigma, \alpha(x_k))\|^p \, d\sigma \right)^{1/p} dt \\
&\le L_f \frac{1}{m(r,\rho)} \sum_{k=1}^{n} \int_{E_k \cap [-r,r]} \rho(t) \left( \int_0^{1} \|\alpha(\sigma + t) - x_k\|^p \, d\sigma \right)^{1/p} dt \\
&+ L_g \frac{1}{m(r,\rho)} \sum_{k=1}^{n} \int_{E_k \cap [-r,r]} \rho(t) \left( \int_0^{1} \|\alpha(\sigma + t) - x_k\|^p \, d\sigma \right)^{1/p} dt \\
&+ \frac{1}{m(r,\rho)} \sum_{k=1}^{n} \int_{E_k \cap [-r,r]} \rho(t) \left( \int_t^{t+1} \|\phi(\sigma, \alpha(x_k))\|^p \, d\sigma \right)^{1/p} dt \\
&< 2\epsilon + \sum_{k=1}^{n} \frac{1}{m(r,\rho)} \int_{-r}^{r} \rho(t) \left( \int_t^{t+1} \|\phi(\sigma, x_k)\|^p \, d\sigma \right)^{1/p} dt.
\end{aligned}$$

Then

$$\frac{1}{m(r,\rho)} \int_{-r}^{r} \rho(t) \left( \int_t^{t+1} \|\phi(\sigma, \alpha(\sigma))\|^p \, d\sigma \right)^{1/p} dt < (n+2)\epsilon \quad (r > r_0).$$

Hence

$$\lim_{r \to \infty} \frac{1}{m(r,\rho)} \int_{-r}^{r} \rho(t) \left( \int_t^{t+1} \|\phi(\sigma, \alpha(\sigma))\|^p \, d\sigma \right)^{1/p} dt = 0.$$

Therefore $H(\cdot) \in \mathcal{B}^{-1}(PAA_0(\mathbb{R}, L^p(0,1;X)))$. It follows that $f(\cdot, h(\cdot)) \in WPP_{\omega ap}S^p(\mathbb{R}, X)$. □

Now, we obtain the existence and uniqueness of weighted pseudo antiperiodic solutions with help of Theorem 2.16 and Theorem 3.2.

**Theorem 3.4.** *Let $\rho \in U_\infty$ and $p > 1$ and $f = g + h \in WPP_{\omega ap}S^p(\mathbb{R} \times X, X)$ be given. Suppose that*

*(H1) There exists $\omega > 0$ such that $f(t+\omega, -x) = -f(t,x)$.*

*(H2) There exist constants $L_f, L_g > 0$ such that*

$$\|f(t,u) - f(t,v)\| \leq L_f\|u - v\|, \quad \|g(t,u) - g(t,v)\| \leq L_g\|u - v\|, \quad t \in \mathbb{R},\, u, v \in X.$$

*(H3) The operator $A$ generates an $\alpha$-resolvent family $\{S_\alpha(t)\}_{t\geq 0}$ such that $\|S_\alpha(t)\| \leq \varphi_\alpha(t)$, for all $t \geq 0$, where $\varphi_\alpha(\cdot) \in L^1(\mathbb{R}_+)$ is nonincreasing such that $\varphi_0 := \sum_{n=0}^{\infty} \varphi_\alpha(n) < \infty$ and $L_f < \|\varphi_\alpha\|^{-1}$.*

*Then the Equation* (3.1) *has a unique mild solution in $WPP_{\omega ap}(\mathbb{R}, X)$.*

*Proof.* Consider the operator $Q : WPP_{\omega ap}(\mathbb{R}, X) \to WPP_{\omega ap}(\mathbb{R}, X)$ defined by

$$Q(u)(t) := \int_{-\infty}^{t} S(t-s)f(s, u(s))\, ds, \quad t \in \mathbb{R}.$$

First, we show that $Q(WPP_{\omega ap}(\mathbb{R}, X)) \subset WPP_{\omega ap}(\mathbb{R}, X)$. Let $u = u_1 + u_2 \in WPP_{\omega ap}(\mathbb{R}, X)$. Then $u_1 \in P_{\omega ap}(\mathbb{R}, X)$ and hence $K := \overline{\{u_1(t) : t \in \mathbb{R}\}}$ is compact. Moreover, it is clear that $u \in WPP_{\omega ap}S^p(\mathbb{R}, X)$ and hence (iii) in Theorem 3.3 is satisfied. From $(H1)$ and $(H2)$ we have that the conditions (i) and (ii) in Theorem 3.3 holds. It follows that $f(\cdot, u(\cdot)) \in WPP_{\omega ap}S^p(\mathbb{R}, X)$. On the other hand, the hypothesis $(H3)$ and Theorem 2.16 imply that $Q(u)(t) \in WPP_{\omega ap}(\mathbb{R}, X)$. Now, if $u, v \in WPP_{\omega ap}(\mathbb{R}, X)$ we have

$$\begin{aligned}
\|Q(u)(t) - Q(v)(t)\|_\infty &= \sup_{t\in\mathbb{R}} \left\| \int_{-\infty}^{t} S(t-s)[f(s,u(s)) - f(s,v(s))]\, ds \right\| \\
&\leq L_f \sup_{t\in\mathbb{R}} \int_0^\infty \|S(s)\| \|u(t-s) - v(t-s)\|\, ds \\
&\leq L_f \|u - v\|_\infty \int_0^\infty \varphi_\alpha(s)\, ds.
\end{aligned}$$

This proves that $Q$ is a contraction, so by the Banach Fixed Point Theorem we conclude that $Q$ has unique fixed point. It follows that $Q(u) = u \in WPP_{\omega ap}(\mathbb{R}, X)$ and is unique. Hence $u$ is the unique mild solution of (3.1). □

We finish this paper with a simple application that no means generality but illustrates how our hypotheses apply.

**Example 3.5.** We put $A = -\varrho$ in $X = \mathbb{R}$, $a(t) = \dfrac{\varrho}{4}\dfrac{t^{\alpha-1}}{\Gamma(\alpha)}$, $\varrho > 0$, $0 < \alpha < 1$, and $f(t,u) = g(t,u) + h(t,u)$ where

$$g(t,u) = \alpha(t)\cos u, \quad h(t,u) = \beta(t)\cos u$$

with $\alpha(t) := \displaystyle\sum_{k=1}^{\infty} \frac{\sin((2k+1)t)}{k^2}$ and $\beta(t) := \dfrac{1}{1+t^2}$.

We note that $g(t+\pi, u) = -g(t,u)$ for all $t$. Furthermore,

$$g(t+\pi, -u) = \cos(-u)\alpha(t+\pi) = \cos(u)[-\alpha(t)] = -\cos(u)\alpha(t) = -g(t,u);$$

Hence $g \in P_{\omega ap}S^p(\mathbb{R} \times X, X)$ with $\omega = \pi$. On the other hand, $h \in PAA_0(\mathbb{R} \times X, X)$ (see [**?**]) and since $PAA_0(\mathbb{R} \times X, X) \subset PAA_0S^p(\mathbb{R} \times X, X)$ then $h \in PAA_0S^p(\mathbb{R} \times X, X)$. It follows that $f \in WPAA_{\omega ap}S^p(\mathbb{R} \times X, X)$ with weight $\rho(t) \equiv 1$. Now,

$$\|g(t,u) - g(t,v)\| \leq 2\sum_{k=1}^{\infty} \frac{1}{k^2};$$

and

$$\|f(t,u) - f(t,v)\| \leq (|\alpha(t)| + |\beta(t)|)\,\|u-v\| \leq \left[\sum_{k=1}^{\infty} \frac{1}{k^2} + 1\right] \|u-v\| = \left(\frac{\pi^2}{6} + 1\right) \|u-v\|.$$

Therefore the functions $g(t,u) := \alpha(t)\cos(u)$, $h(t,u) := \beta(t)\cos u$ verify the hypothesis in Theorem 3.4. Thus, we have that equation (1.2) takes the form

$$D^\alpha u(t) = -\varrho u(t) - \frac{\varrho^2}{4}\int_{-\infty}^{t} \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} u(s)ds + f(t,u(t)), \quad t \in \mathbb{R}. \tag{3.2}$$

From [15, Example 4.17], it follows that $A$ generates an $\alpha$-resolvent family $\{S_\alpha(t)\}_{t\geq 0}$ such that

$$\hat{S}_\alpha(\lambda) = \frac{\lambda^\alpha}{(\lambda^\alpha + 2/\varrho)^2} = \frac{\lambda^{\alpha-\alpha/2}}{(\lambda^\alpha + 2/\varrho)} \cdot \frac{\lambda^{\alpha-\alpha/2}}{(\lambda^\alpha + 2/\varrho)}.$$

Thus, we obtain explicitly

$$S_\alpha(t) = (r * r)(t) \quad t > 0,$$

with $r(t) = t^{\frac{\alpha}{2}-1}E_{\alpha,\frac{\alpha}{2}}(-\frac{\varrho}{2}t^\alpha)$, and where $E_{\alpha,\frac{\alpha}{2}}(\cdot)$ is the Mittag-Leffler function.

Then, by Theorem 3.4, we can conclude that there exists a unique mild solution $u(\cdot) \in WPP_{\omega ap}(\mathbb{R}, X)$ of Eq.(3.2) provided $\|S_\alpha\| < \frac{6}{\pi^2+6}$. We remark that given $0 < \alpha < 1$, we can choose the number $\varrho > 0$ such that $\|S_\alpha\| < \frac{6}{\pi^2+6}$ as in the proof of [15, Lemma 3.9].

## REFERENCES

1. A.R. Aftabizadeh, S. Aizicivici, N.H. Pavel.*On a class of second-order anti-periodic boundary value problems.* J. Math. Anal. Appl. **171** (1992) 301–320.
2. A.R. Aftabizadeh, S. Aizicivici, N.H. Pavel.*Anti-periodic boundary value problems for higher order differential equations in Hilbert spaces.* Nonlinear Anal. **18** (1992) 253–267.
3. N.S. Al-Islam, S.M. Alsulami, T. Diagana. *Existence of weighted pseudo anti-periodic solutions to some non-autonomous differential equations.* Appl. Math. Comput. **218** (2012) 6536–6648.
4. H.L. Chen. *Antiperiodic functions.* J. Comput. Math. 14 (1) (1996) 3239.
5. Y.Q. Chen.*Anti-periodic solutions for semilinear evolution equations.* J. Math. Anal. Appl. **315** (2006) 337–348.
6. T. Diagana. *Existence of pseudo-almost automorphic solutions to some abstract differential equations with Sp-pseudo-almost automorphic coefficients.* Nonlinear Anal. **70** (11) (2009), 3781–3790.
7. T. Diagana. *Weighted pseudo almost periodic functions and applications.* C. R. Acad. Sci. Paris, Ser I **343** (10) (2006), 643–646.
8. T. Diagana, G.M. N'Guérékata and G.M. Mophou. *Existence of weighted pseudo almost periodic solutions to some classes of differential equations with $S^p$-weighted pseudo almost periodic coefficients.* Nonlinear Anal.**72** (2010), 430–438.
9. A. Haraux. *Anti-periodic solutions of some nonlinear evolution equations.* Manuscr. Math. **63** (1989) 479–505.

10. F. Mainardi. *Fractional Calculus and Waves in Linear Viscoelasticity: An Introduction to Mathematical Models,* Imperial College Press, 2010.
11. K. Miller, B. Ross. *An Introduction to the Fractional Calculus and Fractional Differential Equations,* Wiley, New York 1993.
12. I. Mishra, D. Bahuguna. *Weighted pseudo almost automorphic solution of an integro-differential equation, with weighted Stepanov-like pseudo almost automorphic forcing term.* Appl. Math. Comp. **219** (2013), 5345–5355.
13. G.M. N'Guérékata, V. Valmorin. *Antiperiodic solutions for semilinear integrodifferential equations in Banach spaces.* Appl. Math. Comp.**218** (2012), 11118–11124.
14. H. Okoshi. *On the existence of antiperiodic solutions to a nonlinear evolution equation associated with odd subdifferential operators.* J. Funct. Anal. **91** (1990) 246–258.
15. R. Ponce. *Bounded mild solutions to fractional integro-differential equations in Banach spaces.* Semigroup Forum, **87** (2013), 377–392.
16. J. Prüss. *Evolutionary Integral Equations and Applications.* Monographs Math., **87**, Birkhäuser Verlag, 1993.
17. Z. Xia. *Weighted stepanov-like pseudoperiodicity and applications.* Abstr. Appl. Anal. (2014), Art. ID 980869, 14 pp.
18. Z. Xia and M. Fan. *Weighted Stepanov-like pseudo almost automorphy and applications.* Nonlinear Anal. **75** (4) (2012), 2378–2397.

Universidad del Atlántico, Facultad de Ciencias Básicas, Departamento de Matemáticas, Barranquilla, COLOMBIA
*E-mail address*: `edgalp@yahoo.com, edgardoalvarez@mail.uniatlantico.edu.co`

Universidad de Santiago de Chile, Facultad de Ciencia, Departamento de Matemática y Ciencia de la Computación, Casilla 307, Correo 2, Santiago, CHILE
*E-mail address*: `carlos.lizama@usach.cl`

Universidad de Talca, Instituto de Matemática y Física, Casilla 747, Talca-Chile.
*E-mail address*: `rponce@inst-mat.utalca.cl`