Manuscript submitted to
AIMS' Journals
Volume **X**, Number **0X**, XX **200X**

doi:10.3934/xx.xx.xx.xx

pp. **X–XX**

ON WELL-POSEDNESS OF VECTOR-VALUED FRACTIONAL DIFFERENTIAL-DIFFERENCE EQUATIONS

Luciano Abadías

Departamento de Matemáticas, Instituto Universitario de Matemáticas y Aplicaciones
Universidad de Zaragoza
50009 Zaragoza, Spain

Carlos Lizama* Pedro J. Miana and M. Pilar Velasco

Departamento de Matemática y Ciencia de la Computación
Universidad de Santiago de Chile
Las Sophoras 173, Estación Central, Santiago, Chile
Departamento de Matemáticas, Instituto Universitario de Matemáticas y Aplicaciones
Universidad de Zaragoza
50009 Zaragoza, Spain
Escuela Técnica Superior de Ingeniería y Sistemas de Teleomunicación
Universidad Politécnica de Madrid
C/Nikola Tesla, s/n 28031 Madrid, Spain

(Communicated by the associate editor name)

Abstract. We develop an operator-theoretical method for the analysis on well posedness of partial differential-difference equations that can be modeled in the form

$$(*)\left\{\begin{array}{rcl} \Delta^\alpha u(n) & = & Au(n+2)+f(n,u(n)), \quad n\in\mathbb{N}_0,\ 1<\alpha\leq 2;\\ u(0) & = & u_0;\\ u(1) & = & u_1, \end{array}\right.$$

where A is a closed linear operator defined on a Banach space X. Our ideas are inspired on the Poisson distribution as a tool to sampling fractional differential operators into fractional differences. Using our abstract approach, we are able to show existence and uniqueness of solutions for the problem (*) on a distinguished class of weighted Lebesgue spaces of sequences, under mild conditions on sequences of strongly continuous families of bounded operators generated by A, and natural restrictions on the nonlinearity f. Finally we present some original examples to illustrate our results.

Many physical signals, such as electrical voltages produced by a sound or image recording instrument or a measuring device, are essentially continuous-time signals. Computers and related devices operate on a discrete-time axis. Continuous-time signals that are to be processed by such devices therefore first need be converted to discrete-time signals. One way to do it is by sampling.

2010 *Mathematics Subject Classification.* Primary: 35R11; Secondary: 39A14, 47D06.

Key words and phrases. Difference equations, fractional and nonlinear PDE, Poisson distribution, weighted Lebesgue space, well posedness.

C. Lizama has been partially supported by DICYT, Universidad de Santiago de Chile and FONDECYT 1180041. L. Abadias and P. J. Miana have been partially supported by Project MTM2016-77710-P, DGI-FEDER, of the MCYTS, and Project E-64 D.G. Aragón. M. P. Velasco has been partially supported by Project ESP2016-79135-R of the MCYTS..

* Corresponding author: Carlos Lizama.

Let A be a closed linear operator with domain $D(A)$ defined on a Banach space X, and let $u : \mathbb{R}_+ \to X$ be a function that represents a continuous-time signal. Motivated by previous studies of Cuesta, Lubich and Palencia [17, 18], Lizama introduced in [24] a method of sampling for the time-fractional abstract Cauchy problem $D_t^\alpha u(t) = Au(t)$, where D_t^α is the Riemann-Liouville differential operator of order $\alpha > 0$, by means of what he called the Poisson transformation, which is defined by:

$$u(n) := \mathcal{P}(u)(n) = \int_0^\infty p_n(t)u(t)dt, \qquad n \in \mathbb{N}_0, \qquad 0 < \alpha \le 1, \tag{1}$$

where $p_n(t) = e^{-t}\frac{t^n}{n!}$ denotes the Poisson distribution. This transformation has remarkable properties, and some of them have been studied in [24]. A notable property related with the Riemann-Liouville differential operator of order $0 < \alpha < 1$, D_t^α, is the following:

$$\Delta^\alpha u(n) = \int_0^\infty p_{n+1}(t) D_t^\alpha u(t)dt, \quad n \in \mathbb{N}_0, \tag{2}$$

where Δ^α is the operator defined for $m-1 < \alpha < m,\ m \in \mathbb{N}$, by

$$\Delta^\alpha v(n) = \Delta^m\Big(\sum_{j=0}^n k^{m-\alpha}(n-j)v(j)\Big), \quad k^\beta(n) := \frac{\Gamma(\beta+n)}{\Gamma(\beta)\Gamma(n+1)} \quad n \in \mathbb{N}_0, \quad \beta > 0,$$

and Δ^m is the m-th order forward difference operator. See [24] for the proof. Observe that the operator Δ^α is equivalent by translation with the fractional difference operator introduced by Atici and Eloe [9], which has been used recently by some authors in order to obtain several qualitative properties of fractional difference equations concerning stability [16, 33, 36]. However, these properties are given only in the finite dimensional case of X. Also, the kernel k^β appears in many mathematical areas of interest and has some interesting properties. Some of them have been recently discussed in [3]. It is worthwhile to mention that $k^\alpha = \mathcal{P}(g_\alpha)$ where $g_\alpha(t) = \frac{t^{\alpha-1}}{\Gamma(\alpha)}$ for $t > 0$.

Recently, some interesting applications of fractional difference equations have appeared in the literature. In [33] the authors proposed a class of linear fractional difference equations with discrete time delay and impulse effects. They obtained the exact solution via discrete Mittag-Leffler functions and they provided some results about stability and numerical experiments. Some recent applications to image encryption are revisited in [12]. In particular the stability of impulsive fractional difference equations and impulsive effects were introduced jointly with some numerical results which provided support to the analysis. Generalized differential optimization problems driven by the Caputo derivative were treated in [36]. Then existence of weak Carathéodory and a numerical approximation algorithm were also introduced, and some convergence theorems were established. Finally these results were verified by an algorithm in a nonlinear programming problem, see more details in [36].

In [24], it is used the procedure of sampling, by means of (1), in order to prove the existence of a unique solution to the initial value problem

$$\begin{cases} \Delta^\alpha u(n) &= Au(n+1) + f(n), \quad n \in \mathbb{N}_0; \\ u(0) &= u_0 \in X, \end{cases} \tag{3}$$

where $0 < \alpha \le 1$, and it is derived several sufficient conditions for stability in terms of A. For instance, when A is the generator of a C_0-semigroup. However, the study of the case $1 < \alpha \leq 2$ was left as an open problem. We observe that in the scalar case, a characterization of stability has been recently proven (see [15, Theorem 3.1]). It is remarkable that such characterization coincides with the corresponding one, which appears in [24], when $A = \lambda I$ where I denotes the identity operator.

In the paper [25], the author proved a characterization of maximal regularity for the model (3) on vector valued ℓ_p spaces of sequences for $0 < \alpha \leq 1$ and A being a bounded operator. The case $1 < \alpha \leq 2$ was settled in [27]. However, in both cases, a characterization for unbounded operators A is an untreated topic. This problem is important in view of recent advances in numerical methods for Partial Differential Equations [6, 22]. In the work [26], sufficient conditions for the well posedness of the corresponding semilinear problem (3) (i.e. with $f(n)$ replaced by $f(n, u(n))$) were proven but, again, the case $1 < \alpha \leq 2$ was left as an open problem. Note that more recently, a new method to establish the well-posedness in the time scale $\mathbb{Z}$ was proposed in [28]. In such paper, the handling of the full range $\alpha > 0$ was done succesfully.

The general aim of this paper is to continue in this new avenue of research. In particular, we are interested in the following specific problem:

(Q) Given $A : D(A) \subset X \to X$ a closed linear operator defined on a Banach space X, can we establish well-posedness for the linearized version of the abstract problem

$$\begin{cases} \Delta^\alpha u(n) &= Au(n+2) + f(n, u(n)), \quad n \in \mathbb{N}_0; \\ u(0) &= u_0 \in X; \\ u(1) &= u_1 \in X, \end{cases} \tag{4}$$

in case $1 < \alpha \leq 2$?. Can we obtain an explicit representation of the solution in terms of operator families when A is the generator of a C_0-semigroup? Or a cosine/sine family?.

Initial studies on the model (4) when A is a complex or real valued matrix, have only appeared [11, 20]. From a numerical point of view, our analysis refers to schemes that are discretized only in time. The theory of time-discrete fractional equations is a promising tool for several biological and physical applications where the memory effect appears [10, 13]. For example, an application to a model of growth of tumors has been analyzed by Atici and Sengül in [10]. Fractional differences do not only exhibit the advantages of memory effects, as does the continuous case, but they also involve fewer numerical computations. Recent the work of Wu, Baleanu and Xie [34] on fractional chaotic maps reveals this interest. See also the references therein. The study of the chaotic behaviour of the fractional discrete logistic map with delay was recently proposed in an interesting work by Wu and Baleanu [35]. Also, the rate of convergence in the approximation of discrete solutions to continuous solutions has a great importance (see [1, 12]).

The outline of this paper is as follows: In Section 2, we give some background on the definitions to be used, and provide some key properties. For example, we check that

$${}_C\Delta^\alpha u(n) = \Delta^\alpha u(n) - k^{2-\alpha}(n+1)[u(1) - 2u(0)] - k^{2-\alpha}(n+2)u(0), \quad n \in \mathbb{N}_0,$$

and

$$\Delta^\alpha (u * v)(n) = (\Delta^\alpha u * v)(n) + (u(1) - \alpha u(0))v(n+1) + u(0)v(n+2), \qquad n \in \mathbb{N}_0,$$

for $1 < \alpha \leq 2$ and suitable sequences u and v (Theorems 1.5 and 1.6).

In Section 3, we use successfully the preceding definitions and properties to solve the problem (4), firstly, in the homogeneous linear case. In order to do that, we construct a distinguished sequence of bounded and linear operators $\{\mathcal{S}_\alpha(n)\}_{n\in\mathbb{N}_0}$ that solves the homogeneous linear initial value problem

$$\begin{cases} \Delta^\alpha u(n) &= Au(n+2), \quad n\in\mathbb{N}_0, \quad 1<\alpha\leq 2;\\ u(0) &= u_0\in D(A);\\ u(1) &= u_1\in D(A), \end{cases}$$

see Theorem 2.4. This sequence of bounded and linear operators, is defined axiomatically by means of the following two properties:

(i) $\mathcal{S}_\alpha(n)Ax = A\mathcal{S}_\alpha(n)x$ for $n\in\mathbb{N}_0$ and $x\in D(A)$;
(ii) $\mathcal{S}_\alpha(n)x = k^\alpha(n)x + A(k^\alpha * \mathcal{S}_\alpha)(n)x$, for all $n\in\mathbb{N}_0$ and $x\in X$.

In particular, when the operator A is bounded with $\|A\|<1$ we derive an explicit representation of the solution, namely

$$\mathcal{S}_\alpha(n) = \sum_{j=0}^\infty k^{\alpha(j+1)}(n)A^j, \qquad n\in\mathbb{N}_0,$$

(see Proposition 2.2). From a different point of view, this representation can be considered as the discrete counterpart of the Mittag-Leffler function $t^{\alpha-1}E_{\alpha,\alpha}(At^\alpha)$ (when A is a complex number) which interpolates between the exponential and hyperbolic sine function for $1<\alpha<2$. In contrast, when A is unbounded, we give an interesting characterization of the sequence $\{\mathcal{S}_\alpha(n)\}_{n\in\mathbb{N}_0}$ in terms of a series of powers of the bounded operator $(I-A)^{-1}$. See Theorem 2.3.

In Section 4 we study the fully nonlinear problem (4). After to introduce the notion of solution, which is motivated by the representation of the solution in the non-homogeneous linear case (Corollary 2), we consider a distinguished class of vector-valued spaces of weighted sequences, that behaves like

$$\ell_w^\infty(\mathbb{N}_0;X) := \left\{\xi:\mathbb{N}_0\to X \quad / \quad \sup_{n\in\mathbb{N}_0}\frac{\|\xi(n)\|}{nn!}<\infty\right\}.$$

This vector-valued Banach spaces of sequences will play a central role in the development of this section. The main ingredient for the success of our analysis is the observation that the special weight $w(n)=nn!$, that represents the factorial representation of a positive integer, proves to be suitable to find existence and uniqueness of solutions for (4) in the above defined space $\ell_w^\infty(\mathbb{N}_0;X)$ under the hypothesis of only boundedness of the sequence of operators $\{\mathcal{S}_\alpha(n)\}_{n\in\mathbb{N}_0}$. We give two positive results in this direction. The first, requires a Lipschitz type condition on the nonlinear term f and uses the Banach fixed point theorem as main tool. The second, replaces the Lipschitz type condition by compactness of a distinguished set of points that depends on the sequence $\{\mathcal{S}_\alpha(n)\}_{n\in\mathbb{N}_0}$ and f. In this case, we used the Leray-Schauder alternative theorem. See Theorems 3.4 and 3.6, respectively.

In Section 5, we study the Poisson transformation. In order to do this, we first study in detail the Poisson distribution from a functional-analytical point of view (Proposition 4.1). Then, we prove several relations between the continuous and discrete setting, including a generalization of the notable identity (2):

$$\mathcal{P}(D_t^\alpha u)(n+m) = \int_0^\infty p_{n+m}(t)D_t^\alpha u(t)dt = \Delta^\alpha\mathcal{P}(u)(n), \quad n\in\mathbb{N}_0,$$

where $m-1<\alpha\leq m$. See Theorem 4.5 below. This relations are obtained in the context of the Poisson transformation (1) whose main properties are established in Theorem 4.2, extending the pioneer results in [24]. Note that the idea of discretization of the fractional derivative in time was employed in the paper [19] (see also [17] and references therein). We finish this section with concrete examples of Poisson transforms for some well known functions.

Section 6 is devoted to the construction of sequences of operators $\{\mathcal{S}_\alpha(n)\}_{n\in\mathbb{N}_0}$ via subordination by the Poisson transformation of α-resolvent families $(S_\alpha(t))_{t>0}$ generated by A, i.e.,

$$\mathcal{S}_\alpha(n) := \mathcal{P}(S_\alpha)(n), \qquad n \in \mathbb{N}_0,$$

(Theorem 5.3). Then, two interesting examples are provided. The first, corresponds to the discrete counterpart of the Mittag-Leffler function $t^{\alpha-1}E_{\alpha,\alpha}(At^\alpha)$ and, the second, corresponds to a sequence of operators $\{\mathcal{S}_\alpha(n)\}_{n\in\mathbb{N}_0}$ that originates from the generator of a bounded sine family (Examples 5.4 and 5.5, respectively).

A remarkable consequence is Theorem 5.6, which proves existence and uniqueness of solution for the nonlinear problem (4) in the space $\ell^\infty_w(\mathbb{N}_0; X)$ under the hypothesis that A is the generator of a bounded sine family such that the resolvent operator $(\lambda - A)^{-1}$ is a compact operator for some λ large enough.

Finally, Section 7 provides us with several examples and applications of our general theorems, notably concerning the cases where either A is a multiplication operator, $Af(x) = m(x)f(x)$ defined on $L^2(a,b)$ and m a continuous function, see example 6.1, or the second order partial differential operator $\frac{\partial^2}{\partial x^2}$, see example 6.2. We also pay special attention to the case $\alpha = 2$ and to some related problems formatted in a slightly different way than (4). In particular, we consider difference equations in the form

$$\begin{cases} \Delta^2 u(n) &= Bu(n+1) + g(n, u(n)), \quad n \in \mathbb{N}_0;\\ u(0) &= u_0;\\ u(1) &= u_1, \end{cases}$$

where B is a linear operator defined on a Banach space X, see Propositions 6.3 and 6.4.

Notation We denote by $\mathbb{N}_0 := \{0, 1, 2, ...\}$, the set of non-negative integer numbers and X a complex Banach space. We denote by $s(\mathbb{N}_0; X)$ the vectorial space consisting of all vector-valued sequences $u : \mathbb{N}_0 \to X$. We recall that the Z-*transform* of a vector-valued sequence $f \in s(\mathbb{N}_0; X)$, is defined by

$$\widetilde{u}(z) := \sum_{j=0}^{\infty} z^{-j} u(j)$$

where z is a complex number. Note that convergence of the series is given for $|z| > R$ with R sufficiently large.

Recall that the finite convolution $*$ of two sequences $u \in s(\mathbb{N}_0; \mathbb{C})$ and $v \in s(\mathbb{N}_0; X)$ is defined by

$$(u * v)(n) := \sum_{j=0}^{n} u(n-j)v(j), \quad n \in \mathbb{N}_0.$$

It is well known that

$$\widetilde{(u * v)}(z) = \widetilde{u}(z)\widetilde{v}(z), \quad |z| > \max\{R_1, R_2\}, \tag{5}$$

where R_1 and R_2 are the radius of convergence of the Z-transforms of u and v respectively. The Banach space $\ell^1(\mathbb{N}_0; X)$ is the subset of $s(\mathbb{N}_0; X)$ such that $\|u\|_1 :=$

$\sum_{n=0}^{\infty}\|u(n)\|<\infty$; and the Lebesgue space $L^1(\mathbb{R}_+;X)$ is formed by measurable functions $f:\mathbb{R}_+\to X$ such that

$$\|f\|_1 := \int_0^\infty \|f(t)\|dt < \infty.$$

The usual Laplace transform is given by

$$\hat{f}(\lambda) := \int_0^\infty e^{-\lambda t} f(t)dt, \qquad \Re\lambda > 0, \quad f\in L^1(\mathbb{R}_+;X).$$

In the case $X=\mathbb{C}$, the Banach space $L^1(\mathbb{R}_+)$ is, in fact, a Banach algebra with the usual convolution product $*$ given by

$$f*g(t) := \int_0^t f(t-s)g(s)ds, \qquad t\geq 0, \quad f,g\in L^1(\mathbb{R}_+).$$

The same holds in the case of $(\ell^1,*)$. The Banach space $C^{(m)}(\mathbb{R}_+;X)$ is formed for continuous functions which have m-continuous derivatives defined on $\mathbb{R}_+$ with $m\in\mathbb{N}_0$.

Let $S:\mathbb{R}_+\to\mathcal{B}(X)$ be strongly continuous, that is, for all $x\in X$ the map $t\to S(t)x$ is continuous on $\mathbb{R}_+$. We say that a family of bounded and linear operators $\{S(t)\}_{t\geq 0}$ is exponentially bounded if there exist real numbers $M>0$ and $\omega\in\mathbb{R}$ such that

$$\|S(t)\|\leq Me^{\omega t}, \quad t\geq 0.$$

We say that $\{S(t)\}_{t\geq 0}$ is bounded if $\omega=0$. Note that if $\{S(t)\}_{t\geq 0}$ is exponentially bounded then the Laplace transform $\hat{S}(\lambda)x$ exists for all $\Re(\lambda)>\omega$.

1. **Fractional difference operators.** The forward Euler operator $\Delta : s(\mathbb{N}_0;X)\to s(\mathbb{N}_0;X)$ is defined by

$$\Delta u(n) := u(n+1)-u(n), \quad n\in\mathbb{N}_0.$$

For $m\in\mathbb{N}$, we define recursively $\Delta^m : s(\mathbb{N}_0;X)\to s(\mathbb{N}_0;X)$ by $\Delta^1=\Delta$ and

$$\Delta^m := \Delta^{m-1}\circ\Delta.$$

The operator Δ^m is called the m-th order forward difference operator and

$$\Delta^m u(n) = \sum_{j=0}^{m}\binom{m}{j}(-1)^{m-j}u(n+j), \quad n\in\mathbb{N}_0, \tag{6}$$

for $u\in s(\mathbb{N}_0;X)$. We also denote by $\Delta^0=I$, where I is the identity operator.

We define

$$k^\alpha(n) := \frac{\Gamma(\alpha+n)}{\Gamma(\alpha)\Gamma(n+1)}, \quad n\in\mathbb{N}_0, \quad \alpha>0. \tag{7}$$

This sequence, introduced in [24], has appeared in [2], [25] among others, in connection with fractional difference operators. The semigroup property $k^\alpha * k^\beta = k^{\alpha+\beta}$ and the generating formula

$$\sum_{n=0}^{\infty} k^\alpha(n)z^n = \frac{1}{(1-z)^\alpha}, \qquad |z|<1, \tag{8}$$

hold for $\alpha,\beta>0$, see for example [37, Vol. I, p.77].

The following definition of fractional sum (also called Cesàro sum in [37]) has appeared recently in some papers, see for example [2, 24, 25]. It has proven to be useful in the treatment of fractional difference equations. Note that this definition is implicitly included in e.g. [5, 9, 29].

Definition 1.1. [25, Definition 2.5] Let $\alpha > 0$. The α-th fractional sum of a sequence $u : \mathbb{N}_0 \to X$ is defined as follows

$$\Delta^{-\alpha} u(n) := \sum_{j=0}^{n} k^{\alpha}(n-j)u(j) = (k^{\alpha} * u)(n), \quad n \in \mathbb{N}_0. \tag{9}$$

One of the reasons to choose this operator in this paper is because their flexibility to be handled by means of Z-transform methods. Moreover, it has a better behavior for mathematical analysis when we ask, for example, for definitions of fractional sums and differences on subspaces of $s(\mathbb{N}_0; X)$ like e.g. ℓ_p spaces. We notice that, recently, this approach by means of the Z-transform has been followed by other authors, see [15, 16].

The next concept is analogous to the definition of a fractional derivative in the sense of Riemann-Liouville, see [8, 29]. In other words, to a given vector-valued sequence, first fractional summation and then integer difference are applied.

Definition 1.2. [25, Definition 2.7] Let $\alpha \in \mathbb{R}_+ \backslash \mathbb{N}_0$. The fractional difference operator of order α in the sense of Riemann-Liouville, $\Delta^{\alpha} : s(\mathbb{N}_0; X) \to s(\mathbb{N}_0; X)$, is defined by

$$\Delta^{\alpha} u(n) := \Delta^{m}(\Delta^{-(m-\alpha)} u)(n), \quad n \in \mathbb{N}_0,$$

where $m - 1 < \alpha < m$.

Example 1.3. Let $1 < \beta$. Then

$$\Delta k^{\beta}(n) = \frac{\Gamma(\beta+n+1)}{\Gamma(\beta)\Gamma(n+2)} - \frac{\Gamma(\beta+n)}{\Gamma(\beta)\Gamma(n+1)} = \frac{(\beta-1)\Gamma(\beta+n)}{\Gamma(\beta)\Gamma(n+2)} = k^{\beta-1}(n+1),\ n \in \mathbb{N}_0.$$

We iterate m-times with $m \in \mathbb{N}$ to get for $\beta > m$ that

$$\Delta^{m} k^{\beta}(n) = k^{\beta-m}(n+m),\ n \in \mathbb{N}_0.$$

Let $0 < \alpha < \beta$ and $m - 1 < \alpha < m$ for $m \in \mathbb{N}$. By Definition 1.2 and (1.1), we get that

$$\begin{aligned} \Delta^{\alpha} k^{\beta}(n) &= \Delta^{m}(\Delta^{-(m-\alpha)} k^{\beta})(n) = \Delta^{m}(k^{m-\alpha} * k^{\beta})(n) = \Delta^{m}(k^{m-\alpha+\beta}) \\ &= k^{\beta-\alpha}(n+m), \end{aligned}$$

for $n \in \mathbb{N}_0$. This equality extends [24, Corollary 3.6] given for $0 < \alpha < 1$.

Interchanging the order of the operators in the definition of fractional difference in the sense of Riemann-Liouville, and in analogous way as above, we can introduce the notion of fractional difference in the sense of Caputo as follows.

Definition 1.4. [25, Definition 2.8] Let $\alpha \in \mathbb{R}_+ \backslash \mathbb{N}_0$. The fractional difference operator of order α in the sense of Caputo, ${}_C\Delta^{\alpha} : s(\mathbb{N}_0; X) \to s(\mathbb{N}_0; X)$, is defined by

$${}_C\Delta^{\alpha} u(n) := \Delta^{-(m-\alpha)}(\Delta^{m} u)(n), \quad n \in \mathbb{N}_0,$$

where $m - 1 < \alpha < m$.

For further use, we note the following relation between the Caputo and Riemann-Liouville fractional differences of order $1 < \alpha < 2$. The connection between the Caputo and Riemann-Liouville fractional differences of order $0 < \alpha < 1$ is given in [24, Theorem 2.4].

Theorem 1.5. *For each $1<\alpha<2$ and $u\in s(\mathbb{N}_0;X)$, we have*

$$ {}_C\Delta^\alpha u(n)=\Delta^\alpha u(n)-k^{2-\alpha}(n+1)[u(1)-2u(0)]-k^{2-\alpha}(n+2)u(0),\; n\in\mathbb{N}_0. $$

Proof. By definition and (9) we have

$$
\begin{aligned}
\Delta^{-(2-\alpha)}(\Delta^2 u)(n) &= \sum_{j=0}^{n} k^{2-\alpha}(n-j)\Delta^2 u(j)=\sum_{j=0}^{n} k^{2-\alpha}(n-j)u(j+2)\\
&- 2\sum_{j=0}^{n} k^{2-\alpha}(n-j)u(j+1)+\sum_{j=0}^{n} k^{2-\alpha}(n-j)u(j)\\
&= \sum_{j=2}^{n+2} k^{2-\alpha}(n+2-j)u(j)-2\sum_{j=1}^{n+1} k^{2-\alpha}(n+1-j)u(j)\\
&+ \sum_{j=0}^{n} k^{2-\alpha}(n-j)u(j)\\
&= \sum_{j=0}^{n+2} k^{2-\alpha}(n+2-j)u(j)\\
&- 2\sum_{j=0}^{n+1} k^{2-\alpha}(n+1-j)u(j)+\sum_{j=0}^{n} k^{2-\alpha}(n-j)u(j)\\
&- k^{2-\alpha}(n+2)u(0)-k^{2-\alpha}(n+1)u(1)+2k^{2-\alpha}(n+1)u(0)\\
&= \Delta^2(\Delta^{-(2-\alpha)}u)(n)-k^{2-\alpha}(n+1)(u(1)\\
&- 2u(0))-k^{2-\alpha}(n+2)u(0),
\end{aligned}
$$

and so we obtain the desired result. □

We also have the following property for the Riemann-Liouville fractional difference of the convolution.

Theorem 1.6. *Let $1<\alpha\le 2$, $u\in s(\mathbb{N}_0;\mathbb{C})$ and $v\in s(\mathbb{N}_0;X)$. Then, for each $n\in\mathbb{N}_0$ the following identity holds*

$$ \Delta^\alpha(u*v)(n)=(\Delta^\alpha u*v)(n)+(u(1)-\alpha u(0))v(n+1)+u(0)v(n+2). $$

Proof. For each $n \in \mathbb{N}_0$,

$$\begin{aligned}
\Delta^\alpha(u*v)(n) &= \Delta^{-(2-\alpha)}(u*v)(n+2) - 2\Delta^{-(2-\alpha)}(u*v)(n+1) \\
&+ \Delta^{-(2-\alpha)}(u*v)(n) \\
&= \sum_{j=0}^{n+2}(k^{2-\alpha}*u)(n+2-j)v(j) - 2\sum_{j=0}^{n+1}(k^{2-\alpha}*u)(n+1-j)v(j) \\
&+ \sum_{j=0}^{n}(k^{2-\alpha}*u)(n-j)v(j) \\
&= \sum_{j=0}^{n}(k^{2-\alpha}*u)(n+2-j)v(j) - 2\sum_{j=0}^{n}(k^{2-\alpha}*u)(n+1-j)v(j) \\
&+ \sum_{j=0}^{n}(k^{2-\alpha}*u)(n-j)v(j) + (k^{2-\alpha}*u)(1)v(n+1) \\
&+ (k^{2-\alpha}*u)(0)v(n+2) - 2(k^{2-\alpha}*u)(0)v(n+1) \\
&= \sum_{j=0}^{n}\Delta^2(k^{2-\alpha}*u)(n-j)v(j) + (k^{2-\alpha}(0)u(1) + k^{2-\alpha}(1)u(0))v(n+1) \\
&+ (k^{2-\alpha}(0)u(0))v(n+2) - 2(k^{2-\alpha}(0)u(0))v(n+1) \\
&= \sum_{j=0}^{n}\Delta^\alpha u(n-j)v(j) + (u(1) + (2-\alpha)u(0))v(n+1) + u(0)v(n+2) \\
&- 2u(0)v(n+1) \\
&= (\Delta^\alpha u * v)(n) + (u(1) - \alpha u(0))v(n+1) + u(0)v(n+2),
\end{aligned}$$

proving the claim. □

We notice that for $0 < \alpha \le 1$ the above property reads

$$\Delta^\alpha(u*v)(n) = (\Delta^\alpha u * v)(n) + u(0)v(n+1), \quad n \in \mathbb{N}_0.$$

It has been proved only recently in [25, Lemma 3.6].

2. **Linear fractional difference equations on Banach spaces.** Let A be a closed linear operator defined on a Banach space X. In this section we study the problem

$$\begin{cases} \Delta^\alpha u(n) &= Au(n+2) + f(n), \quad n \in \mathbb{N}_0, \quad 1 < \alpha \le 2; \\ u(0) &= u_0; \\ u(1) &= u_1. \end{cases} \tag{10}$$

Following [24], we say that a vector valued sequence $u \in s(\mathbb{N}_0; X)$ is a solution of (10) if $u(n) \in D(A)$ for all $n \in \mathbb{N}_0$ and u satisfies (10).

We will use the notion of discrete α-resolvent family introduced in [2, Definition 3.1] to obtain the solution of the problem (10). Note that the knowledge of the abstract properties of this family of bounded operators provide insights on the qualitative behavior of the solutions of fractional difference equations.

Definition 2.1. Let $\alpha > 0$ and A be a closed linear operator with domain $D(A)$ defined on a Banach space X. An operator-valued sequence $\{\mathcal{S}_\alpha(n)\}_{n\in\mathbb{N}_0} \subset \mathcal{B}(X)$

is called a discrete α-resolvent family generated by A if it satisfies the following conditions

(i) For each $x \in X$, $\mathcal{S}_\alpha(t)x \in D(A)$ and $\mathcal{S}_\alpha(n)Ax = A\mathcal{S}_\alpha(n)x$ for $n \in \mathbb{N}_0$ and $x \in D(A)$;
(ii) $\mathcal{S}_\alpha(n)x = k^\alpha(n)x + A(k^\alpha * \mathcal{S}_\alpha)(n)x$, for all $n \in \mathbb{N}_0$ and $x \in X$.

The family $\{\mathcal{S}_\alpha(n)\}_{n\in\mathbb{N}_0}$ is said bounded if $\|\mathcal{S}\|_\infty := \sup_{n\in\mathbb{N}_0} \|\mathcal{S}_\alpha(n)\| < \infty$.

An explicit representation of a discrete α-resolvent family generated by a bounded operator A with $\|A\| < 1$ is given in the following proposition.

Proposition 2.2. Let $\alpha > 0$ and $A \in \mathcal{B}(X)$, with $\|A\| < 1$. Then the operator A generates a discrete α-resolvent family $\{\mathcal{S}_\alpha(n)\}_{n\in\mathbb{N}_0}$ given by

$$\mathcal{S}_\alpha(n) = \sum_{j=0}^{\infty} k^{\alpha(j+1)}(n)A^j, \qquad n \in \mathbb{N}_0.$$

Proof. Since $k^\alpha(n) = \frac{n^{\alpha-1}}{\Gamma(\alpha)}(1 + O(\frac{1}{n}))$, for $n \in \mathbb{N}$ (see for example [37, Vol. I, (1.18)]), then the series is convergent for $\|A\| < 1$. Take $x \in X$ and $n \in \mathbb{N}_0$. Then we get that

$$\begin{aligned} A(k^\alpha * \mathcal{S}_\alpha)(n)x &= A\sum_{j=0}^{n} k^\alpha(n-j)\mathcal{S}_\alpha(j)x = A\sum_{j=0}^{n} k^\alpha(n-j)\sum_{i=0}^{\infty} k^{\alpha(i+1)}(j)A^i x \\ &= \sum_{i=0}^{\infty} A^{i+1}x\sum_{j=0}^{n} k^\alpha(n-j)k^{\alpha(i+1)}(j) = \sum_{i=0}^{\infty} k^{\alpha(i+2)}(n)A^{i+1}x, \end{aligned}$$

where we have applied the semigroup property of the kernel k^α. Then we obtain

$$k^\alpha(n)x + A(k^\alpha * \mathcal{S}_\alpha)(n)x = \sum_{i=0}^{\infty} k^{\alpha(i+1)}(n)A^i x = \mathcal{S}_\alpha(n)x, \qquad n \in \mathbb{N}_0,$$

and we conclude the proof. □

For $\alpha > 0$ fixed and each $n \in \mathbb{N}$ the sequence $\{\beta_{\alpha,n}(j)\}_{j=1,\ldots,n}$ was introduced in [2, Section 3.1] as follows:

For $n = 1$,

$$\beta_{\alpha,1}(1) = k^\alpha(1) = \alpha.$$

For $n = 2$,

$$\begin{aligned} \beta_{\alpha,2}(1) &= k^\alpha(2) - k^\alpha(1)\beta_{\alpha,1}(1) = k^\alpha(2) - (k^\alpha(1))^2, \\ \beta_{\alpha,2}(2) &= k^\alpha(1)\beta_{\alpha,1}(1) = (k^\alpha(1))^2 = \alpha^2. \end{aligned}$$

For $n = 3$,

$$\begin{aligned} \beta_{\alpha,3}(1) &= k^\alpha(3) - k^\alpha(2)\beta_{\alpha,1}(1) - k^\alpha(1)\beta_{\alpha,2}(1) = k^\alpha(3) - 2k^\alpha(2)k^\alpha(1) + (k^\alpha(1))^3, \\ \beta_{\alpha,3}(2) &= k^\alpha(2)\beta_{\alpha,1}(1) + k^\alpha(1)\beta_{\alpha,2}(1) - k^\alpha(1)\beta_{\alpha,2}(2) = 2k^\alpha(2)k^\alpha(1) - 2\,(k^\alpha(1))^3, \\ \beta_{\alpha,3}(3) &= k^\alpha(1)\beta_{\alpha,2}(2) = (k^\alpha(1))^3 = \alpha^3. \end{aligned}$$

For $n \geq 4$,

$$
\begin{aligned}
\beta_{\alpha,n}(1) &= k^{\alpha}(n) - \sum_{j=1}^{n-1} k^{\alpha}(n-j)\beta_{\alpha,j}(1), \\
\beta_{\alpha,n}(l) &= \sum_{j=l-1}^{n-1} k^{\alpha}(n-j)\beta_{\alpha,j}(l-1) - \sum_{j=l}^{n-1} k^{\alpha}(n-j)\beta_{\alpha,j}(l) \quad \text{for} \quad 2 \leq l \leq n-1, \\
\beta_{\alpha,n}(n) &= k^{\alpha}(1)\beta_{\alpha,n-1}(n-1) = (k^{\alpha}(1))^n = \alpha^n
\end{aligned}
$$

In case that A is closed, but not necessarily bounded, the authors in [2, Theorem 3.2] proven that given $\{\mathcal{S}_\alpha(n)\}_{n\in\mathbb{N}_0} \subset \mathcal{B}(X)$ a discrete α-resolvent family generated by A, then $1 \in \rho(A)$ and $\mathcal{S}_\alpha(0) = (I-A)^{-1}$; $\mathcal{S}_\alpha(0)x \in D(A)$ and $\mathcal{S}_\alpha(n)x \in D(A^2)$ for all $n \in \mathbb{N}$, and $x \in X$; and

$$
\mathcal{S}_\alpha(n)x = \sum_{j=1}^{n} \beta_{\alpha,n}(j)(I-A)^{-(j+1)}x, \quad n \in \mathbb{N}, \quad x \in X.
$$

The last equality provides an explicit representation of discrete α-resolvent families in terms of a bounded linear operators which is, in fact, a characterization of this family of operators as the next theorem shows.

Theorem 2.3. *Let $\lambda, \alpha > 0$, $(A, D(A))$ be a closed operator on the Banach space X and $\{\mathcal{S}_\alpha(n)\}_{n\in\mathbb{N}_0} \subset \mathcal{B}(X)$ be a sequence of bounded operators. Then the following conditions are equivalent.*

(i) The family $\{\mathcal{S}_\alpha(n)\}_{n\in\mathbb{N}_0} \subset \mathcal{B}(X)$ is a discrete α-resolvent family generated by A.
(ii) $1 \in \rho(A)$, the operator $\mathcal{S}_\alpha(0) = (I-A)^{-1}$ and

$$
\mathcal{S}_\alpha(n)x = \sum_{j=1}^{n} \beta_{\alpha,n}(j)(I-A)^{-(j+1)}x, \quad n \in \mathbb{N}, \quad x \in X.
$$

If there exists $\lambda_0 > 0$ such that $\sup_{n\in\mathbb{N}_0} \lambda_0^{-n}\|\mathcal{S}_\alpha(n)\| < \infty$, both equations are equivalent to

(iii) $\left(\frac{\lambda-1}{\lambda}\right)^{\alpha} \in \rho(A)$ and

$$
\left(\left(\frac{\lambda-1}{\lambda}\right)^{\alpha} - A\right)^{-1} x = \sum_{n=0}^{\infty} \lambda^{-n}\mathcal{S}_\alpha(n)x, \qquad x \in X, |\lambda| > \max\{\lambda_0, 1\}. \tag{11}
$$

Proof. The condition (i) implies the condition (ii) is given in [2, Theorem 3.2]. Now we suppose that the condition (ii) holds. Then $\mathcal{S}_\alpha(n)x \in D(A)$ for any $x \in X$ and

$n \in \mathbb{N}_0$. For $n \in \mathbb{N}$ and $x \in X$ we have that

$$\begin{aligned}(I-A)\mathcal{S}_\alpha(n)x = \sum_{j=1}^{n} \beta_{\alpha,n}(j)(I-A)^{-j}x = (k^\alpha(n) \\ - \sum_{i=1}^{n-1} k^\alpha(n-i)\beta_{\alpha,i}(1))(I-A)^{-1}x \\ + \sum_{j=2}^{n-1}(\sum_{i=j-1}^{n-1} k^\alpha(n-i)\beta_{\alpha,i}(j-1) - \sum_{i=j}^{n-1} k^\alpha(n-i)\beta_{\alpha,i}(j))(I-A)^{-j}x \\ + k^\alpha(1)\beta_{\alpha,n-1}(n-1)(I-A)^{-n}x \\ = k^\alpha(n)(I-A)^{-1}x \\ + \sum_{j=1}^{n-1}\sum_{i=j}^{n-1} k^\alpha(n-i)\beta_{\alpha,i}(j)((I-A)^{-(j+1)} - (I-A)^{-j})x.\end{aligned}$$

Applying the identities $(I-A)^{-1} - I = A(I-A)^{-1}$ and $\mathcal{S}_\alpha(0) = (I-A)^{-1}$ we get that

$$\begin{aligned}(I-A)\mathcal{S}_\alpha(n)x &= k^\alpha(n)(I + A\mathcal{S}_\alpha(0))x \\ &\quad + A\sum_{i=1}^{n-1} k^\alpha(n-i)\sum_{j=1}^{i}\beta_{\alpha,i}(j)(I-A)^{-(j+1)}x \\ &= k^\alpha(n)(I + A\mathcal{S}_\alpha(0))x + A\sum_{i=1}^{n-1} k^\alpha(n-i)\mathcal{S}_\alpha(i)x,\end{aligned}$$

and clearly it follows that $\mathcal{S}_\alpha(n)x = k^\alpha(n)x + A(k^\alpha * \mathcal{S}_\alpha)(n)x$ for $n \in \mathbb{N}$. The case $n = 0$ is a simple check.

Finally we prove that if there exists $\lambda_0 > 0$ such that $\sup_{n\in\mathbb{N}_0} \lambda_0^{-n}\|\mathcal{S}_\alpha(n)\| < \infty$, (iii) is equivalent to (i). Assume that $\{\mathcal{S}_\alpha(n)\}_{n\in\mathbb{N}_0} \subset \mathcal{B}(X)$ is a discrete α-resolvent family generated by A, then applying Z-transform we get for $|\lambda| > \max\{\lambda_0, 1\}$ that

$$\begin{aligned}\widetilde{\mathcal{S}_\alpha}(\lambda)x &= \sum_{j=0}^{\infty} \lambda^{-j}\mathcal{S}_\alpha(j)x = \widetilde{k^\alpha}(\lambda)x + A\widetilde{k^\alpha}(\lambda)\widetilde{\mathcal{S}_\alpha}(\lambda)x \\ &= \left(\frac{\lambda}{\lambda-1}\right)^\alpha x + \left(\frac{\lambda}{\lambda-1}\right)^\alpha A\widetilde{\mathcal{S}_\alpha}(\lambda)x, \quad x \in X,\end{aligned}$$

and

$$\widetilde{\mathcal{S}_\alpha}(\lambda)x = \left(\frac{\lambda}{\lambda-1}\right)^\alpha x + \left(\frac{\lambda}{\lambda-1}\right)^\alpha \widetilde{\mathcal{S}_\alpha}(\lambda)Ax, \quad x \in D(A),$$

where we have used that (5) and (8). Thus the operator $\left(\frac{\lambda-1}{\lambda}\right)^\alpha - A$ is invertible, and we get (11). Conversely, let $|\lambda|, |\mu| > \max\{\lambda_0, 1\}$ and $x \in D(A)$, then there exists $y \in X$ such that $x = \left(\left(\frac{\mu-1}{\mu}\right)^\alpha - A\right)^{-1} y$. Using that $\left(\left(\frac{\lambda-1}{\lambda}\right)^\alpha - A\right)^{-1}$ and $\left(\left(\frac{\mu-1}{\mu}\right)^\alpha - A\right)^{-1}$ are bounded operators and commute, and A is closed we have that

$$\begin{aligned}\widetilde{\mathcal{S}_\alpha}(\lambda)x &= \widetilde{\mathcal{S}_\alpha}(\lambda)\left(\left(\frac{\mu-1}{\mu}\right)^\alpha - A\right)^{-1} y\\ &= \left(\left(\frac{\mu-1}{\mu}\right)^\alpha - A\right)^{-1}\left(\left(\frac{\lambda-1}{\lambda}\right)^\alpha - A\right)^{-1} y\\ &= \sum_{n=0}^{\infty}\lambda^{-n}\left(\left(\frac{\mu-1}{\mu}\right)^\alpha - A\right)^{-1}\mathcal{S}_\alpha(n)\left(\left(\frac{\mu-1}{\mu}\right)^\alpha - A\right)x.\end{aligned}$$

The uniqueness of Z-transform proves that

$$\mathcal{S}_\alpha(n)x = \left(\left(\frac{\mu-1}{\mu}\right)^\alpha - A\right)^{-1}\mathcal{S}_\alpha(n)\left(\left(\frac{\mu-1}{\mu}\right)^\alpha - A\right)x.$$

Then we have $\mathcal{S}_\alpha(n)x \in D(A)$, and therefore $A\mathcal{S}_\alpha(n)x = \mathcal{S}_\alpha(n)Ax$ for all $x \in X$. Finally note that for $|\lambda| > \max\{\lambda_0, 1\}$ and $x \in D(A)$ we have using (8) that

$$\begin{aligned}\widetilde{k^\alpha}(\lambda)x &= \widetilde{k^\alpha}(\lambda)\widetilde{\mathcal{S}_\alpha}(\lambda)\left(\left(\frac{\lambda-1}{\lambda}\right)^\alpha - A\right)x\\ &= \widetilde{\mathcal{S}_\alpha}(\lambda)x - (\widetilde{k^\alpha * \mathcal{S}_\alpha})(\lambda)Ax,\end{aligned}$$

and by the uniqueness of Z-transform we get the result. □

A nice consequence of Theorem 2.3 is the following result about sums of combinatorial numbers which seems to be new.

Corollary 1. *Take $\alpha > 0$, $n \in \mathbb{N}$ and $\{\beta_{\alpha,n}(j)\}_{j=1,\dots,n}$ defined as above. Then*

(i) $\displaystyle\sum_{j=1}^{n}\frac{\beta_{\alpha,n}(j)}{(1-\lambda)^{j+1}} = \sum_{l=0}^{\infty}\lambda^l\frac{\Gamma(\alpha(l+1)+n)}{\Gamma(\alpha(l+1))\Gamma(n+1)}$, *for* $|\lambda| < 1$.

(ii) $\displaystyle\sum_{j=1}^{n}\beta_{\alpha,n}(j)\frac{\Gamma(l+1+j)}{\Gamma(l+1)\Gamma(j+1)} = \frac{\Gamma(\alpha(l+1)+n)}{\Gamma(\alpha(l+1))\Gamma(n+1)}$, *for* $l \in \mathbb{N}$.

Proof. (i) We take $|\lambda| < 1$, then using Proposition 2.2 and Theorem 2.3 in the scalar case we have that

$$\sum_{j=1}^{n}\frac{\beta_{\alpha,n}(j)}{(1-\lambda)^{j+1}} = \sum_{l=0}^{\infty}\lambda^l k^{\alpha(l+1)}(n) = \sum_{l=0}^{\infty}\lambda^l\frac{\Gamma(\alpha(l+1)+n)}{\Gamma(\alpha(l+1))\Gamma(n+1)}, \qquad n \in \mathbb{N}.$$

(ii) Let $|\lambda| < 1$, then

$$\sum_{j=1}^{n}\frac{\beta_{\alpha,n}(j)}{(1-\lambda)^{j+1}} = \sum_{l=0}^{\infty}\lambda^l\sum_{j=1}^{n}\beta_{\alpha,n}(j)\frac{\Gamma(l+1+j)}{\Gamma(l+1)\Gamma(j+1)}, \qquad n \in \mathbb{N},$$

where we have applied that $\displaystyle\frac{1}{(1-\lambda)^{j+1}} = \sum_{l=0}^{\infty}\lambda^l\frac{\Gamma(l+1+j)}{\Gamma(l+1)\Gamma(j+1)}$. Then we apply (i) to get the result. □

Our main result in this section is the following theorem.

Theorem 2.4. *Let $1 < \alpha \leq 2$. Suppose that A is the generator of a discrete α-resolvent family $\{\mathcal{S}_\alpha(n)\}_{n\in\mathbb{N}_0}$ on a Banach space X. Then the fractional difference equation*

$$\Delta^\alpha u(n) = Au(n+2), \quad n \in \mathbb{N}_0, \tag{12}$$

with initial conditions $u(0) = u_0 \in D(A)$ and $u(1) = u_1 \in D(A)$ admits the unique solution

$$u(n) = \mathcal{S}_\alpha(n)(I - A)u(0) - \alpha\mathcal{S}_\alpha(n-1)u(0) + \mathcal{S}_\alpha(n-1)(I - A)u(1), \ n \in \mathbb{N}_0.$$

Proof. Convolving the identity given in Definition 2.1(ii) by $k^{2-\alpha}$, we obtain

$$(k^{2-\alpha} * \mathcal{S}_\alpha)(n)x = (k^{2-\alpha} * k^\alpha)(n)x + A(k^{2-\alpha} * k^\alpha * \mathcal{S}_\alpha)(n)x, \quad n \in \mathbb{N}_0.$$

Using the semigroup property for the kernels k^α we have

$$(k^{2-\alpha} * \mathcal{S}_\alpha)(n)x = k^2(n)x + A(k^2 * \mathcal{S}_\alpha)(n)x, \quad n \in \mathbb{N}_0.$$

This is equivalent, by definition of fractional sum and convolution, to the following identity

$$\Delta^{-(2-\alpha)}\mathcal{S}_\alpha(n)x = k^2(n)x + A\sum_{j=0}^{n} k^2(n-j)\mathcal{S}_\alpha(j)x, \quad n \in \mathbb{N}_0.$$

Therefore, we get using $\Delta^2 k^2(j) = 0$ for $j \in \mathbb{N}_0$ that

$$\begin{aligned}
\Delta^2 \circ \Delta^{-(2-\alpha)}\mathcal{S}_\alpha(n)x &= \Delta^2 k^2(n)x + A\Delta^2 \sum_{j=0}^{n} k^2(n-j)\mathcal{S}_\alpha(j)x \\
&= A\Big[\sum_{j=0}^{n+2} k^2(j)\mathcal{S}_\alpha(n+2-j)x - 2\sum_{j=0}^{n+1} k^2(j)\mathcal{S}_\alpha(n+1-j)x \\
&\quad + \sum_{j=0}^{n} k^2(j)\mathcal{S}_\alpha(n-j)x\Big] \\
&= A\Big[\sum_{j=2}^{n+2} k^2(j)\mathcal{S}_\alpha(n+2-j)x - 2\sum_{j=1}^{n+1} k^2(j)\mathcal{S}_\alpha(n+1-j)x \\
&\quad + \sum_{j=0}^{n} k^2(j)\mathcal{S}_\alpha(n-j)x + \mathcal{S}_\alpha(n+2)k^2(0)x \\
&\quad + \mathcal{S}_\alpha(n+1)k^2(1)x - 2\mathcal{S}_\alpha(n+1)k^2(0)x\Big] \\
&= A\Big[\sum_{j=0}^{n} k^2(j+2)\mathcal{S}_\alpha(n-j)x - 2\sum_{j=0}^{n} k^2(j+1)\mathcal{S}_\alpha(n-j)x \\
&\quad + \sum_{j=0}^{n} k^2(j)\mathcal{S}_\alpha(n-j)x + \mathcal{S}_\alpha(n+2)x\Big]
\end{aligned}$$

for all $n \in \mathbb{N}_0$. We note that the left hand side in the above identity corresponds to the fractional difference of order $\alpha \in (0,2)$ in the sense of Riemann-Liouville. Therefore, we obtain

$$\Delta^\alpha \mathcal{S}_\alpha(n)x = A\mathcal{S}_\alpha(n+2)x, \tag{13}$$

for all $n \in \mathbb{N}_0$ and all $x \in X$. Define $u(n)$ as

$$u(n) := \mathcal{S}_\alpha(n)(I - A)u(0) - \alpha\mathcal{S}_\alpha(n-1)u(0) + \mathcal{S}_\alpha(n-1)(I - A)u(1), \quad n \in \mathbb{N}_0.$$

It then follows from (13) that u solves (12). Finally, from the identities

$$\mathcal{S}_\alpha(0)x = k^\alpha(0)x + A(k^\alpha * \mathcal{S}_\alpha)(0)x = x + Ak^\alpha(0)\mathcal{S}_\alpha(0)x = x + A\mathcal{S}_\alpha(0)x,$$

and

$$\mathcal{S}_\alpha(1)x = k^\alpha(1)x + A(k^\alpha * \mathcal{S}_\alpha)(1)x = \alpha\mathcal{S}_\alpha(0)x + A\mathcal{S}_\alpha(1)x,$$

which follow from Definition 2.1 (ii), we obtain $u(0) = \mathcal{S}_\alpha(0)(I-A)u_0 = u_0$ and $u(1) = \mathcal{S}_\alpha(1)(I-A)u_0 - \alpha\mathcal{S}_\alpha(0)u_0 + \mathcal{S}_\alpha(0)(I-A)u_1 = u_1$, and we conclude the proof. □

In the non homogeneous case, we derive the following result.

Corollary 2. *Let $1 < \alpha \le 2$. Suppose that A is the generator of a discrete α-resolvent family $\{\mathcal{S}_\alpha(n)\}_{n\in\mathbb{N}_0}$ on a Banach space X and f be a vector-valued sequence. The fractional difference equation*

$$\Delta^\alpha u(n) = Au(n+2) + f(n), \quad n \in \mathbb{N}_0,\ 1 < \alpha \le 2, \tag{14}$$

with initial conditions $u(0) = u_0 \in D(A)$ and $u(1) = u_1 \in D(A)$, admits the unique solution

$$u(n) = \mathcal{S}_\alpha(n)(I-A)u(0) - \alpha\mathcal{S}_\alpha(n-1)u(0) + \mathcal{S}_\alpha(n-1)(I-A)u(1) + (\mathcal{S}_\alpha * f)(n-2),$$

for all $n \ge 2$.

Proof. Indeed, by Theorem 2.4 and Theorem 1.6 we have $u(n) \in D(A)$ for all $n \ge 2$ and

$$\begin{aligned}
\Delta^\alpha u(n) &= \Delta^\alpha(\mathcal{S}_\alpha(n)(I-A)u(0) - \alpha\mathcal{S}_\alpha(n-1)u(0) + \mathcal{S}_\alpha(n-1)(I-A)u(1)) \\
&\quad + \Delta^\alpha(\mathcal{S}_\alpha * f)(n-2) \\
&= A(\mathcal{S}_\alpha(n+2)(I-A)u(0) - \alpha\mathcal{S}_\alpha(n+1)u(0) + \mathcal{S}_\alpha(n)(I-A)u(1)) \\
&\quad + (\Delta^\alpha\mathcal{S}_\alpha * f)(n-2) + (\mathcal{S}_\alpha(1) - \alpha\mathcal{S}_\alpha(0))f(n-1) + \mathcal{S}_\alpha(0)f(n) \\
&= Au(n+2) - (A\mathcal{S}_\alpha * f)(n) + (\Delta^\alpha\mathcal{S}_\alpha * f)(n-2) \\
&\quad + (\mathcal{S}_\alpha(1) - \alpha\mathcal{S}_\alpha(0))f(n-1) + \mathcal{S}_\alpha(0)f(n).
\end{aligned}$$

From (13) it follows $(\Delta^\alpha\mathcal{S}_\alpha * f)(n-2) = (A\mathcal{S}_\alpha * f)(n) - A\mathcal{S}_\alpha(1)(n-1) - A\mathcal{S}_\alpha(0)(n)$, and hence we obtain

$$\begin{aligned}
\Delta^\alpha u(n) &= Au(n+2) + (I-A)\mathcal{S}_\alpha(0)f(n) + ((I-A)\mathcal{S}_\alpha(1) - \alpha\mathcal{S}_\alpha(0))f(n-1) \\
&= Au(n+2) + f(n),
\end{aligned}$$

where we have used that $\mathcal{S}_\alpha(0) = (I-A)^{-1}$ and $(I+A)\mathcal{S}_\alpha(1) = \alpha\mathcal{S}_\alpha(0)$. For $n=0$ and $n=1$ it is a simple check, using the same above arguments, that u is solution of (14). □

3. **Non-linear fractional difference equations on Banach spaces.** Let A be a closed linear operator defined on a Banach space X. In this section we study the non linear problem

$$\begin{cases}
\Delta^\alpha u(n) &= Au(n+2) + f(n, u(n)), \quad n \in \mathbb{N}_0,\ 1 < \alpha \le 2; \\
u(0) &= 0; \\
u(1) &= 0.
\end{cases} \tag{15}$$

For the case $0 < \alpha \le 1$ see the reference [26]. The following definition is motivated by Corollary 2. In particular, it shows consistence with the problem (15).

Definition 3.1. Under the assumption that the operator A is the generator of a discrete α-resolvent family $\{\mathcal{S}_\alpha(n)\}_{n\in\mathbb{N}_0}$ on a Banach space X, we say that $u:\mathbb{N}_0\to X$ is a solution of the non-linear problem (15) if u satisfies

$$u(n)=\sum_{k=0}^{n-2}\mathcal{S}_\alpha(n-2-k)f(k,u(k)),\ n=2,3,4,...$$

The next concept of admissibility is one of the keys ingredients for the estimates that we will use in the proofs of our main results on existence of solutions to (15).

Definition 3.2. We say that a sequence $h:\mathbb{N}_0\to(0,\infty)$ is an admissible weight if

$$\lim_{n\to\infty}h(n)=\infty \quad \text{and} \quad \lim_{n\to\infty}\frac{1}{h(n)}\sum_{k=0}^{n-2}h(k)=0.$$

Example 3.3. The sequence $h(n)=nn!$, that represents the factorial number system, is an admissible weight function, since by [**?**, formula 33 p.598], we have $\sum_{k=1}^{n}kk!=(n+1)!-1.$

For each admissible weight sequence h, we consider the vector-valued weighted space

$$\ell_h^\infty(\mathbb{N}_0;X)=\{\xi:\mathbb{N}_0\to X\mid \|\xi\|_h<\infty\},$$

where the norm $\|\quad\|_h$ is defined by $\|\xi\|_h:=\sup_{n\in\mathbb{N}_0}\dfrac{\|\xi(n)\|}{h(n)}$.

The following is our first positive result on existence of solutions for the problem (15). It uses a Lipschitz type condition.

Theorem 3.4. *Let h be an admissible weight and define*

$$H:=\sup_{n\in\{2,3,\dots\}}\frac{1}{h(n)}\sum_{k=0}^{n-2}h(k).$$

Let A be the generator of a bounded discrete α-resolvent family $\{\mathcal{S}_\alpha(n)\}_{n\in\mathbb{N}_0}$ on a Banach space X for some $1<\alpha\le 2$, and let $f:\mathbb{N}_0\times X\to X$ be given and verifying the following hypothesis:

- *(F) $f(k,0)\neq 0$ for all $k\in\mathbb{N}_0$ and there exists a positive sequence $a\in\ell^1(\mathbb{N}_0)$ and constants $c\ge 0$, $b>0$ such that $\|f(k,x)\|\le a(k)(c\|x\|+b)$ for all $k\in\mathbb{N}_0$ and $x\in X$.*
- *(L) The function f satisfies a Lipschitz condition in $x\in X$ uniformly in $k\in\mathbb{N}_0$, that is, there exists a constant $L>0$ such that $\|f(k,x)-f(k,y)\|\le L\|x-y\|$, for all $x,y\in X$, $k\in\mathbb{N}_0$, with $L<(\|\mathcal{S}_\alpha\|_\infty H)^{-1}$.*

Then the problem (15) has an unique solution in $\ell_h^\infty(\mathbb{N}_0;X)$.

Proof. Let us define the operator $G:\ell_h^\infty(\mathbb{N}_0;X)\to\ell_h^\infty(\mathbb{N}_0;X)$ given by

$$Gu(n)=\sum_{k=0}^{n-2}\mathcal{S}_\alpha(n-2-k)f(k,u(k)),\quad n\ge 2.$$

First, we show that G is well defined: Let $u \in \ell_h^\infty(\mathbb{N}_0;X)$ be given. By using the assumption (F) and the boundedness of $\{\mathcal{S}_\alpha(n)\}_{n\in\mathbb{N}_0}$ we get that,

$$\|Gu(n)\| \leq \sum_{k=0}^{n-2}\|\mathcal{S}_\alpha(n-2-k)\|\|f(k,u(k))\| \leq \sum_{k=0}^{n-2}\|\mathcal{S}_\alpha(n-2-k)\|a(k)[c\|u(k)\|+b]$$
$$\leq c\|\mathcal{S}_\alpha\|_\infty\|a\|_\infty\|u\|_h\sum_{k=0}^{n-2}h(k)+b\|\mathcal{S}_\alpha\|_\infty\|a\|_1$$

for each $n \in \mathbb{N}_0$. Hence,

$$\frac{\|Gu(n)\|}{h(n)} \leq c\|\mathcal{S}_\alpha\|_\infty\|a\|_\infty\|u\|_h\frac{1}{h(n)}\sum_{k=0}^{n-2}h(k)+\frac{1}{h(n)}b\|\mathcal{S}_\alpha\|_\infty\|a\|_1.$$

and since h is an admissible weight, the above inequality proves that $Gu \in \ell_h^\infty(\mathbb{N}_0;X)$. We next prove that G is a contraction on $\ell_h^\infty(\mathbb{N}_0;X)$. Indeed, let $u,v \in \ell_h^\infty(\mathbb{N}_0;X)$ be given. Then, for each $n \in \mathbb{N}_0$,

$$\begin{aligned}\|Gu(n)-Gv(n)\| &\leq \sum_{k=0}^{n-2}\|\mathcal{S}_\alpha(n-2-k)\|\|f(k,u(k))-f(k,v(k))\|\\ &\leq \|\mathcal{S}_\alpha\|_\infty\sum_{k=0}^{n-2}\|f(k,u(k))-f(k,v(k))\|\\ &\leq \|\mathcal{S}_\alpha\|_\infty\sum_{k=0}^{n-2}L\|u(k)-v(k)\| \leq \|\mathcal{S}_\alpha\|_\infty L\|u-v\|_h\sum_{k=0}^{n-2}h(k),\end{aligned}$$

where we have used the assumption (L). Therefore

$$\frac{\|Gu(n)-Gv(n)\|}{h(n)} \leq \|\mathcal{S}_\alpha\|_\infty L\|u-v\|_h\frac{1}{h(n)}\sum_{k=0}^{n-2}h(k),$$

and consequently

$$\|Gu-Gv\|_h \leq \|\mathcal{S}_\alpha\|_\infty HL\|u-v\|_h,$$

with $\|\mathcal{S}_\alpha\|_\infty HL<1$. Then, G has a unique fixed point in $\ell_h^\infty(\mathbb{N}_0;X)$, by the Banach fixed point theorem. □

The next Lemma provide a necessary tool for the use of the Schauder's fixed point theorem, needed in the second main result on existence and uniqueness of solutions to (15).

Lemma 3.5. *Let h be an admissible weight and $U \subset \ell_h^\infty(\mathbb{N}_0;X)$ such that:*

(a) The set $H_n(U)=\left\{\frac{u(n)}{h(n)} : u\in U\right\}$ is relatively compact in X, for all $n\in\mathbb{N}_0$.

(b) $\lim_{n\to\infty}\frac{1}{h(n)}\sup_{u\in U}\|u(n)\|=0$, that is, for each $\varepsilon>0$, there are $N>0$ such that $\frac{\|u(n)\|}{h(n)}<\varepsilon$, for each $n\geq N$ and for all $u\in U$.

Then U is relatively compact in $\ell_h^\infty(\mathbb{N}_0;X)$.

Proof. Let $\{u_m\}_m$ be a sequence in U, then by (a) for $n\in\mathbb{N}_0$ there is a convergent subsequence $\{u_{m_j}\}_j\subset\{u_m\}_m$ such that $\lim_{j\to\infty}\frac{u_{m_j}(n)}{h(n)}=a(n)$, that is, for each $\varepsilon>0$

there exists $N(n,\varepsilon) > 0$ such that $\|\frac{u_{m_j}(n)}{h(n)} - a(n)\| < \varepsilon$ for all $j \geq N(n,\varepsilon)$. Let $\varepsilon > 0$ and N the value of the assumption (b). If we consider $N^* := \min_{0\leq n<N} N(n,\varepsilon)$, then for $j,k \geq N^*$ we have

$$\sup_{0\leq n<N} \frac{\|u_{m_j}(n) - u_{m_k}(n)\|}{h(n)} \leq \sup_{0\leq n<N} \|\frac{u_{m_j}(n)}{h(n)} - a(n)\| + \sup_{0\leq n<N} \|\frac{u_{m_k}(n)}{h(n)} - a(n)\| < \varepsilon/2 + \varepsilon/2 = \varepsilon,$$

and also

$$\sup_{n\geq N} \frac{\|u_{m_j}(n) - u_{m_k}(n)\|}{h(n)} \leq \sup_{n\geq N} \frac{\|u_{m_j}(n)\|}{h(n)} + \sup_{n\geq N} \frac{\|u_{m_k}(n)\|}{h(n)} < \varepsilon/2 + \varepsilon/2 = \varepsilon.$$

Consequently,

$$\|u_{m_j} - u_{m_k}\|_h = \sup_{n\in\mathbb{N}_0} \frac{\|u_{m_j}(n) - u_{m_k}(n)\|}{h(n)} < \varepsilon,$$

therefore $\{u_{m_j}\}_j$ is a Cauchy subsequence in $\ell_h^\infty(\mathbb{N}_0;X)$ which finishes the proof. □

For $f : \mathbb{N}_0 \times X \to X$ we recall that the Nemytskii operator $\mathcal{N}_f : \ell_h^\infty(\mathbb{N}_0;X) \to \ell_h^\infty(\mathbb{N}_0;X)$ is defined by

$$\mathcal{N}_f(u)(n) := f(n,u(n)), \quad n \in \mathbb{N}_0.$$

The next theorem is the second main result for this section. It gives one useful criteria for the existence of solutions without use of Lipchitz type conditions.

Theorem 3.6. *Let h be an admissible weight function. Let A be the generator of a bounded discrete α-resolvent family $\{\mathcal{S}_\alpha(n)\}_{n\in\mathbb{N}_0}$ on a Banach space X for some $1 < \alpha \leq 2$, and $f : \mathbb{N}_0 \times X \to X$. Suppose that the condition (F) and following conditions are satisfied:*

(i) The Nemytskii operator is continuous in $\ell_h^\infty(\mathbb{N}_0;X)$, that is, for each $\varepsilon > 0$, there is $\delta > 0$ such that for all $u, v \in \ell_h^\infty(\mathbb{N}_0;X)$, $\|u - v\|_h < \delta$ implies that $\|\mathcal{N}_f(u) - \mathcal{N}_f(v)\|_h < \varepsilon$.

(ii) For all $a \in \mathbb{N}_0$ and $\sigma > 0$, the set $\{\mathcal{S}_\alpha(n)f(k,x) : 0 \leq k \leq a, \|x\| \leq \sigma\}$ is relatively compact in X for all $n \in \mathbb{N}_0$.

Then the problem (15) has an unique solution in $\ell_h^\infty(\mathbb{N};X)$.

Proof. Let us define the operator $G : \ell_h^\infty(\mathbb{N}_0;X) \to \ell_h^\infty(\mathbb{N}_0;X)$ given by

$$Gu(n) = \sum_{k=0}^{n-2} \mathcal{S}_\alpha(n-2-k)f(k,u(k)), \quad n \geq 2.$$

To prove that G has a fixed point in $\ell_h^\infty(\mathbb{N}_0)$, we will use Leray-Schauder alternative theorem. We verify that the conditions of the theorem are satisfied:

- G is well defined: It follows from condition (F) and was proved in the first part of the proof of Theorem 3.4.

- G is continuous: Let $\varepsilon > 0$ and $u, v \in \ell^\infty_h(\mathbb{N}_0)$. Then, for each $n \in \mathbb{N}_0$,

$$\begin{aligned}\|Gu(n) - Gv(n)\| &\leq \sum_{k=0}^{n-2} \|\mathcal{S}_\alpha(n-2-k)\| \|f(k,u(k)) - f(k,v(k))\| \\ &\leq \|\mathcal{S}_\alpha\|_\infty \sum_{k=0}^{n-2} \|f(k,u(k)) - f(k,v(k))\| \\ &\leq \|\mathcal{S}_\alpha\|_\infty \|\mathcal{N}_f(u) - \mathcal{N}_f(v)\|_h \sum_{k=0}^{n-2} h(k).\end{aligned}$$

  Therefore

$$\frac{\|Gu(n) - Gv(n)\|}{h(n)} \leq \|\mathcal{S}_\alpha\|_\infty \|\mathcal{N}_f(u) - \mathcal{N}_f(v)\|_h \frac{1}{h(n)} \sum_{k=0}^{n-2} h(k).$$

  Hence, by the assumption (ii) and admissibility of h we obtain $\|Gu - Gv\|_h < \varepsilon$.
- G is compact: For $R > 0$ given, let $B_R(\ell^\infty_h(\mathbb{N}_0; X)) := \{w \in \ell^\infty_h(\mathbb{N}_0; X) : \|w\|_h < R\}$. To prove that $V := G(B_R(\ell^\infty_h(\mathbb{N}_0; X)))$ is relatively compact, we will use Lemma 3.5. We check that the conditions in such Lemma are satisfied:

  (a) Let $u \in B_R(\ell^\infty_h(\mathbb{N}_0; X))$ and $v = Gu$. We have

$$v(n) = Gu(n) = \sum_{k=0}^{n-2} \mathcal{S}_\alpha(n-2-k) f(k, u(k)) = \sum_{k=0}^{n-2} \mathcal{S}_\alpha(k) f(n-2-k, u(n-2-k)),$$

  and then,

$$\frac{v(n)}{h(n)} = \frac{n-1}{h(n)} \left(\frac{1}{n-1} \sum_{k=0}^{n-2} \mathcal{S}_\alpha(k) f(n-2-k, u(n-2-k)) \right).$$

  Therefore $\frac{v(n)}{h(n)} \in \frac{n-1}{h(n)} co(K_n)$, where $co(K_n)$ denotes the convex hull of K_n for the set

$$K_n = \bigcup_{k=0}^{n-2} \{\mathcal{S}_\alpha(k) f(\xi, x) : \xi \in \{0, 1, 2, \dots, n-2\}, \|x\| \leq R\}, \quad n \in \mathbb{N}_0.$$

  Note that each set K_n is relatively compact by the assumption (ii). From the inclusions $H_n(V) = \left\{ \frac{v(n)}{h(n)} : v \in V \right\} \subseteq \frac{n-1}{h(n)} co(K_n) \subseteq \frac{n-1}{h(n)} co(\overline{K_n})$, we conclude that the set $H_n(V)$ is relatively compact in X, for all $n \in \mathbb{N}_0$.

  (b) Let $u \in B_R(\ell^\infty_h(\mathbb{N}_0; X))$ and $v = Gu$. Using condition (F), for each $n \in \{2, 3, ...\}$ we have
  For each $n \in \mathbb{N}_0$, we have

$$\begin{aligned}\frac{\|v(n)\|}{h(n)} &\leq \frac{1}{h(n)} \sum_{k=0}^{n-2} \|\mathcal{S}_\alpha(n-2-k)\| \|f(k, u(k))\| \\ &\leq c\|\mathcal{S}_\alpha\|_\infty \|a\|_\infty \|u\|_h \frac{1}{h(n)} \sum_{k=0}^{n-2} h(k) + \frac{1}{h(n)} b\|\mathcal{S}_\alpha\|_\infty \|a\|_1 \\ &\leq c\|\mathcal{S}_\alpha\|_\infty \|a\|_\infty R \frac{1}{h(n)} \sum_{k=0}^{n-2} h(k) + \frac{1}{h(n)} b\|\mathcal{S}_\alpha\|_\infty \|a\|_1,\end{aligned}$$

then the admissibility of h implies $\lim_{n\to\infty} \frac{\|v(n)\|}{h(n)} = 0$ independently of $u \in B_R(\ell_h^\infty(\mathbb{N}_0;X))$. Therefore, $V = G(B_R(\ell_h^\infty(\mathbb{N}_0;X)))$ is relatively compact in $\ell_h^\infty(\mathbb{N}_0;X)$ by Lemma 3.5 and we conclude that G is a compact operator.

- The set $U := \{u \in \ell_h^\infty(\mathbb{N}_0;X) : u = \gamma G u, \gamma \in (0,1)\}$ is bounded: In fact, let us consider $u \in \ell_h^\infty(\mathbb{N}_0;X)$ such that $u = \gamma G u$, $\gamma \in (0,1)$. Again by condition (F),

$$\|u(n)\| = \|\gamma G u(n)\| \leq \sum_{k=0}^{n-2} \|\mathcal{S}_\alpha(n-2-k)\| \|f(k,u(k))\|$$
$$\leq c\|\mathcal{S}_\alpha\|_\infty \|a\|_\infty \|u\|_h \sum_{k=0}^{n-2} h(k) + b\|\mathcal{S}_\infty\| \|a\|_1$$

Then for each $n \in \{2,3,...\}$ we have

$$\frac{\|u(n)\|}{h(n)} \leq c\|\mathcal{S}_\alpha\|_\infty \|a\|_\infty \|u\|_h \frac{1}{h(n)} \sum_{k=0}^{n-2} h(k) + \frac{b}{h(n)} \|\mathcal{S}_\infty\| \|a\|_1.$$

Since h is admissible, we deduce that U is a bounded set in $\ell_h^\infty(\mathbb{N}_0;X)$.

Finally, by using the Leray-Schauder alternative theorem, we conclude that G has a fixed point $u \in \ell_h^\infty(\mathbb{N}_0;X)$. □

4. **The Poisson transformation of fractional difference operators.**

For each $n \in \mathbb{N}_0$, the Poisson distribution is defined by

$$p_n(t) := e^{-t} \frac{t^n}{n!}, \quad t \geq 0. \tag{16}$$

The Poisson distribution arises in connection with classical Poisson processes and semigroups of functions; note that it is also called fractional integral semigroup in [30, Theorem 2.6]. In this section we study in detail this sequence of functions (Proposition 4.1), the Poisson transformation (considered deeply in Theorem 4.2) and give their connection with fractional difference and differential operators in Theorem 4.5.

Proposition 4.1. Let $n \in \mathbb{N}_0$ and $(p_n)_{n\geq 0}$ given by (16). Then

(i) For $t \geq 0$, the inequality $p_n(t) \geq 0$ holds, $\int_0^\infty p_n(t)dt = 1$, and

$$\int_0^\infty p_n(t) p_m(t) dt = \frac{1}{2^{n+m+1}} \frac{(n+m)!}{n!m!}, \qquad n,m \in \mathbb{N}_0.$$

(ii) The semigroup property $p_n * p_m = p_{n+m}$ holds for $n,m \in \mathbb{N}_0$.

(iii) Given $t \geq 0$, then

$$(p_{(\cdot)})(t) * p_{(\cdot)}(t))(n) = 2^n e^{-t} p_n(t), \qquad n \in \mathbb{N}_0.$$

(iv) For $m,n \in \mathbb{N}_0$, we have $\Delta^m p_n = (-1)^m p_{n+m}^{(m)}$.

(v) The Z-transform and the Laplace transform are given by

$$\widetilde{p_{(\cdot)}(t)}(z) = e^{-t(1-\frac{1}{z})}, \qquad z \neq 0, \quad t > 0;$$
$$\widehat{p_n}(\lambda) = \frac{1}{(\lambda+1)^{n+1}}, \qquad \Re\lambda > -1, \quad n \in \mathbb{N}_0.$$

Proof. The proof of (i) and (ii) is straightforward, and also may be found in [30, Theorem 2.6]. To show (iii), note that

$$(p_{(\cdot)})(t) * p_{(\cdot)}(t))(n) = e^{-2t}\frac{t^n}{n!}\sum_{j=0}^{n}\frac{n!}{j!(n-j)!} = 2^n e^{-t}p_n(t),$$

for $n \in \mathbb{N}_0$ and $t \geq 0$. Now we get that

$$(p_{n+1})'(t) = -e^{-t}\frac{t^{n+1}}{(n+1)!} + e^{-t}\frac{t^n}{n!} = -\Delta p_n(t),$$

and we iterate to obtain the equality $\Delta^m p_n = (-1)^m p_{n+m}^{(m)}$ for $m, n \in \mathbb{N}_0$. Finally the Z-transform and the Laplace transform of $(p_n)_{n\geq 0}$ are easily obtained. □

Now we introduce an integral transform using the Poisson distribution as integral kernel. Some of their properties are inspired in results included in [24, Section 3] in particular a remarkable connection between the vector-valued Z-transform and the vector-valued Laplace transform, Theorem 4.2 (ii).

Theorem 4.2. *Let $\psi \in L^1(\mathbb{R}_+; X)$ and we define $(\mathcal{P}\psi) \in s(\mathbb{N}_0; X)$ by*

$$(\mathcal{P}\psi)(n) := \int_0^\infty p_n(t)\psi(t)dt, \quad n \in \mathbb{N}_0. \tag{17}$$

Then the following properties hold.

(i) The map $\mathcal{P}$ defines a bounded linear operator from $L^1(\mathbb{R}_+; X)$ to $\ell^1(\mathbb{N}_0; X)$ and $\|\mathcal{P}\| = 1$.

(ii) For $\psi \in L^1(\mathbb{R}_+; X)$, we have that

$$\mathcal{P}(\psi)(n) = \frac{(-1)^n}{n!}\left[\widehat{\psi}(\lambda)\right]^{(n)}|_{\lambda=1}, \qquad n \in \mathbb{N}_0.$$

In particular the map $\mathcal{P}$ is inyective.

(iii) We have that $\widetilde{(\mathcal{P}\psi)}(z) = \widehat{\psi}(1 - 1/z)$, for $|z| > 1$.

(iv) For $a \in L^1(\mathbb{R}_+)$ and $\psi \in L^1(\mathbb{R}_+; X)$ then $\mathcal{P}(a * \psi) = \mathcal{P}(a) * \mathcal{P}(\psi)$.

(v) If there are constants $M > 0$ and $\omega \geq 0$ such that $\|\psi(t)\| \leq Me^{-\omega t}$ for a.e. $t \geq 0$ then $\|\mathcal{P}(\psi)(n)\| \leq \dfrac{M}{(1+\omega)^{n+1}}$ for all $n \in \mathbb{N}_0$. In particular if ψ is bounded then $\{\mathcal{P}(\psi)(n)$ for $n \in \mathbb{N}_0$ is well-defined by (17) and $\{\mathcal{P}(\psi)(n)\}_{n\in\mathbb{N}_0}$ is bounded.

(vi) Let X be a Banach lattice and $\psi(t) \geq 0$ for a.e. $t \geq 0$, then $\mathcal{P}(\psi)(n) \geq 0$ for $n \in \mathbb{N}_0$.

(vii) Suppose that $\{S(t)\}_{t\geq 0} \subset \mathcal{B}(X)$ is a uniformly bounded family of operators. If $\{S(t)\}_{t\geq 0}$ is compact then $\{\mathcal{P}(S)(n)\}_{n\in\mathbb{N}_0}$ is compact.

Proof. To prove (i) is enough to observe that

$$\|\mathcal{P}\psi\|_1 \leq \sum_{n=0}^\infty \int_0^\infty p_n(t)\|\psi(t)\|dt = \int_0^\infty \sum_{n=0}^\infty \frac{t^n}{n!}e^{-t}\|\psi(t)\|dt = \int_0^\infty \|\psi(t)\|dt = \|\psi\|_1,$$

for $\psi \in L^1(\mathbb{R}_+; X)$. Take $0 \neq x \in X$ and define $(e_\lambda \otimes x)(t) := e^{-\lambda t}x$ for $t, \lambda > 0$. Note that $e_\lambda \otimes x \in L^1(\mathbb{R}_+; X)$ and $\|e_\lambda \otimes x\|_1 = \frac{1}{\lambda}\|x\|$ for $\lambda > 0$. It is straightforward to check that

$$\mathcal{P}(e_\lambda \otimes x)(n) = \frac{1}{(1+\lambda)^{n+1}}x, \qquad \lambda > 0, \quad n \in \mathbb{N}_0,$$

and $\|\mathcal{P}(e_\lambda\otimes x)\|_1=\frac{1}{\lambda}\|x\|$ for $\lambda>0$. We conclude that $\|\mathcal{P}\|=1$.

By properties of Laplace transform, see for example [7, Theorem 1.5.1], we have that

$$\mathcal{P}(\psi)(n)=\frac{(-1)^n}{n!}\left[\widehat{\psi}(\lambda)\right]^{(n)}|_{\lambda=1},\qquad \psi\in L^1(\mathbb{R}_+;X).$$

Now take $\psi\in L^1(\mathbb{R}_+;X)$ such that $\mathcal{P}(\psi)(n)=0$ for all $n\in\mathbb{N}_0$. Then we also get that $\left[\widehat{\psi}(\lambda)\right]^{(n)}|_{\lambda=1}=0$ for $n\in\mathbb{N}_0$. Since $\widehat{\psi}$ is an holomorphic function, we conclude that $\widehat{\psi}=0$ and then $\psi=0$ where we apply that the Laplace transform is injective, see for example [7, Theorem 1.7.3].

Part (iii) is proved following similar ideas than in [24, Theorem 3.1]. For (iv) note that because $a\in L^1(\mathbb{R}_+)$ and $\psi\in L^1(\mathbb{R}_+;X)$ we have $a*\psi\in L^1(\mathbb{R}_+;X)$ and

$$\begin{aligned}\mathcal{P}(a*\psi)(n) &= \int_0^\infty \frac{t^n}{n!}e^{-t}\int_0^t a(s)\psi(t-s)dsdt\\ &= \int_0^\infty a(s)e^{-s}\int_0^\infty \frac{(s+u)^n}{n!}e^{-u}\psi(u)duds\\ &= \sum_{j=0}^\infty \int_0^\infty a(s)e^{-s}\frac{s^j}{j!}ds\int_0^\infty \frac{u^{n-j}}{(n-j)!}e^{-u}\psi(u)du=\left(\mathcal{P}(a)*\mathcal{P}(\psi)\right)(n),\end{aligned}$$

for $n\in\mathbb{N}_0$. Assertion (v) and (vi) are easily checked and assertion (vii) is obtained from [32, Corollary 2.3]. □

We check Poisson transforms of some known functions in the next example. Note that, in fact, the Poisson transform can be extended to other sets than $L^1(\mathbb{R}_+;X)$, for example, $\mathcal{P}(f)(n)$ is well-defined for measurable functions f such that

$$\text{ess sup}_{t\geq 0}\|e^{\omega t}f(t)\|<\infty.$$

Also the identity given in Theorem 4.2 (iii) holds for the Dirac distribution δ_t for $t>0$, see Proposition 4.1 (v).

Definition 4.3. The map $\mathcal{P}:L^1(\mathbb{R}_+;X)\to\ell^1(\mathbb{N}_0;X)$ defined by (17) is called the Poisson transformation.

Example 4.4. (i) Note that $\mathcal{P}(e_\lambda)(n)=\dfrac{1}{(\lambda+1)^{n+1}}$ for $n\in\mathbb{N}_0$, where $e_\lambda(t):=e^{-\lambda t}$ for $t>0$.

(ii) By Proposition 4.1(i),

$$\mathcal{P}(p_m)(n)=\frac{1}{2^{n+m+1}}\frac{(n+m)!}{n!m!}=\frac{1}{2^{n+m+1}}k^{m+1}(n),\qquad n,m\in\mathbb{N}_0,$$

where the kernel k^α is defined in (7).

(iii) For $\alpha>0$, define

$$g_\alpha(t):=\begin{cases}\dfrac{t^{\alpha-1}}{\Gamma(\alpha)}, & t>0;\\ 0, & t=0.\end{cases}$$

Then the identity $\mathcal{P}(g_\alpha)=k^\alpha$ holds, see more details in [24, Example 3.3].

(iv) The Mittag-Leffler function is an entire function defined by

$$E_{\alpha,\beta}(z):=\sum_{k=0}^\infty\frac{z^k}{\Gamma(\alpha k+\beta)},\quad \alpha,\beta>0,\qquad z\in\mathbb{C},$$

see for example [14, Section 1.3]. Now take $\lambda \in \mathbb{C}$ such that $|\lambda| < 1$ and $s_{\alpha,\beta}(t) := t^{\beta-1}E_{\alpha,\beta}(\lambda t^{\alpha})$, for $t > 0$. Then

$$\mathcal{P}(s_{\alpha,\beta})(n) = \sum_{k=0}^{\infty} \frac{\lambda^k}{n!\Gamma(\alpha k + \beta)} \int_0^{\infty} t^{n+\alpha k+\beta-1} e^{-t} dt = \sum_{k=0}^{\infty} \frac{\lambda^k \Gamma(n + \alpha k + \beta)}{n!\Gamma(\alpha k + \beta)}, n \in \mathbb{N}_0,$$

which extends the result [24, Theorem 4.7] proved for $\beta = 1$. In the particular case $\beta = \alpha$, we get that

$$\mathcal{P}(s_{\alpha,\alpha})(n) = \sum_{j=0}^{\infty} \frac{\lambda^j \Gamma(n + \alpha(j+1))}{n!\Gamma(\alpha(j+1))} = \sum_{j=0}^{\infty} \lambda^j k^{\alpha(j+1)}(n), \qquad n \in \mathbb{N}_0.$$

Now we are interested to establish a notable relation between the discrete and continuous fractional concepts in the sense of Riemann-Liouville. In order to give our next result, we recall that the Riemann-Liouville fractional integral of order $\alpha > 0$, of a locally integrable function $u : [0, \infty) \to X$ is given by:

$$I_t^{\alpha} u(t) := (g_{\alpha} * u)(t) := \int_0^t g_{\alpha}(t-s)u(s)ds, \qquad t > 0.$$

The Riemann-Liouville fractional derivative of order α, for $m - 1 < \alpha < m$, $m \in \mathbb{N}$, is defined by

$$D_t^{\alpha} u(t) := \frac{d^m}{dt^m} \int_0^t g_{m-\alpha}(t-s)u(s)ds = \frac{d^m}{dt^m}(g_{m-\alpha} * u)(t), \qquad t > 0, \tag{18}$$

for $u \in C^{(m)}(\mathbb{R}_+; X)$, see for example [14, Section 1.2] and [13, Section 1.3]. Compare these definitions with Definitions 1.1 and 1.2.

Theorem 4.5. *Let $m \in \mathbb{N}$ and $m - 1 < \alpha \leq m$. Take $u \in C^{(m)}(\mathbb{R}_+; X)$ such that $e_{-\omega}u^{(m)}$ is integrable for some $0 < \omega < 1$. Then we have*

$$\mathcal{P}(D_t^{\alpha} u)(n+m) = \int_0^{\infty} p_{n+m}(t) D_t^{\alpha} u(t) dt = \Delta^{\alpha} \mathcal{P}(u)(n), \quad n \in \mathbb{N}_0.$$

Proof. Set $n \in \mathbb{N}_0$, $m \in \mathbb{N}$ and $u \in C^{(m)}(\mathbb{R}_+; X)$ such that $e_{-\omega}u^{(m)}$ is integrable for some $0 < \omega < 1$. We integrate by parts m-times to get

$$\begin{aligned} \mathcal{P}(D_t^m u)(n+m) &= \int_0^{\infty} p_{n+m}(t) D_t^m u(t) dt = \cdots = (-1)^m \int_0^{\infty} p_{n+m}^{(m)}(t) u(t) dt \\ &= \int_0^{\infty} \Delta^m p_n(t) u(t) dt = \Delta^m \mathcal{P}(u)(n) \end{aligned}$$

where we have applied Proposition 4.1 (iv).

Now consider $m - 1 < \alpha < m$. By the definition of Riemann-Liouville fractional derivative, see formula (18), we have that

$$\begin{aligned} \mathcal{P}(D_t^{\alpha} u)(n+m) &= \int_0^{\infty} p_{n+m}(t) D_t^{\alpha} u(t) dt = \int_0^{\infty} p_{n+m}(t) \frac{d^m}{dt^m}(g_{m-\alpha} * u)(t) dt \\ &= \Delta^m \mathcal{P}(g_{m-\alpha} * u)(n) = \Delta^m \left(k^{m-\alpha} * \mathcal{P}(u)\right)(n) \\ &= \Delta^m \left(\Delta^{-(m-\alpha)} \mathcal{P}(u)\right)(n) = \Delta^{\alpha} \mathcal{P}(u)(n), \end{aligned}$$

where we have applied Theorem 4.2 (iv), Example 4.4 (iii) and Definition 1.1. $\square$

5. **Discrete α-resolvent families via Poisson subordination.** We recall the following concept (see [4, 23] and references therein).

Definition 5.1. Let A be a closed linear operator with domain $D(A)$ defined on a Banach space X and $\alpha > 0$. We call A the generator of an α-resolvent family if there exists $\omega \geq 0$ and a strongly continuous function $S_\alpha : [0,\infty) \to \mathcal{B}(X)$ (respectively $S_\alpha : (0,\infty) \to \mathcal{B}(X)$ in case $0 < \alpha < 1$) such that $\{\lambda^\alpha : \mathrm{Re}(\lambda) > \omega\} \subset \rho(A)$, the resolvent set of A, and

$$(\lambda^\alpha - A)^{-1}x = \int_0^\infty e^{-\lambda t} S_\alpha(t)x dt, \quad \mathrm{Re}(\lambda) > \omega, \quad x \in X.$$

In this case, $S_\alpha(t)$ is called the α-resolvent family generated by A.

By the uniqueness theorem for the Laplace transform, a 1-resolvent family is the same as a C_0-semigroup, while a 2-resolvent family corresponds to a strongly continuous sine family. See for example [7] and the references therein for an overview on these concepts. Some properties of $(S_\alpha(t))_{t>0}$ are included in the following Lemma. For a proof, see for example [23].

Lemma 5.2. *Let $\alpha > 0$. The following properties hold:*

(i) $\lim_{t\to 0^+} \dfrac{S_\alpha(t)x}{g_\alpha(t)} = x$ *for all* $x \in X$.
(ii) $S_\alpha(t)D(A) \subset D(A)$ *and* $AS_\alpha(t)x = S_\alpha(t)Ax$ *for all* $x \in D(A)$, $t > 0$.
(iii) For all $x \in D(A) : S_\alpha(t)x = g_\alpha(t)x + \int_0^t g_\alpha(t-s)AS_\alpha(s)xds$, $t > 0$.
(iv) For all $x \in X : (g_\alpha * S_\alpha)(t)x \in D(A)$ *and*

$$S_\alpha(t)x = g_\alpha(t)x + A\int_0^t g_\alpha(t-s)S_\alpha(s)xds, \ \ t > 0.$$

The next theorem is the main result of this section.

Theorem 5.3. *Suppose that A is the generator of an α-resolvent family $(S_\alpha(t))_{t>0}$ on a Banach space X, of exponential bound less than 1. Then A is the generator of a discrete α-resolvent family $(\mathcal{S}_\alpha(n))_{n\in\mathbb{N}_0}$ defined by*

$$\mathcal{S}_\alpha(n) := \mathcal{P}(S_\alpha)(n), \qquad n \in \mathbb{N}_0.$$

Proof. Take $x \in D(A)$. Since $(A, D(A))$ is a closed operator and the condition in Lemma 5.2(ii) we have that

$$\mathcal{S}_\alpha(n)Ax = \int_0^\infty p_n(t)S(t)Axdt = \int_0^\infty p_n(t)AS(t)xdt = A\mathcal{S}_\alpha(n)x.$$

From the identity

$$S_\alpha(t)x = g_\alpha(t)x + A\int_0^t g_\alpha(t-s)S_\alpha(s)xds, \quad t > 0,$$

valid for all $x \in X$, we obtain

$$\mathcal{S}_\alpha(n)x = \mathcal{P}(S_\alpha)(n)x = \mathcal{P}(g_\alpha)(n) + A\mathcal{P}(g_\alpha * S_\alpha)(n)x = k^\alpha(n) + A\left(k^\alpha * \mathcal{S}_\alpha\right)(n)x,$$

where we have applied Example 4.4(iii) and Theorem 4.2 (iv) and the second condition in Definition 2.1. The theorem is proved. $\square$

Example 5.4. Consider the Mittag-Leffler function $E_{\alpha,\beta}$ studied in Example 4.4 (iv). Suppose that A is a bounded operator on the Banach space X. It follows from Definition 5.1 that

$$S_\alpha(t) = t^{\alpha-1}E_{\alpha,\alpha}(At^\alpha), \quad t > 0,\, \alpha > 0$$

is the α-resolvent family generated by A. If $\|A\| < 1$, then

$$\mathcal{S}_\alpha(n)x := \int_0^\infty e^{-t}\frac{t^n}{n!}t^{\alpha-1}E_{\alpha,\alpha}(At^\alpha)x dt = \sum_{k=0}^\infty \frac{\Gamma(\alpha(k+1)+n)}{\Gamma(\alpha(k+1))\Gamma(n+1)}A^k x, \quad n \in \mathbb{N}_0,$$

for $x \in X$. Compare with Proposition 2.2.

Example 5.5. Suppose that A is the generator of a bounded sine family $(S(t))_{t>0}$ on X. Then A is the generator of a bounded α-resolvent family $(S_\alpha(t))_{t>0}$ on X for $1 < \alpha < 2$ given by

$$S_\alpha(t)x = \int_0^\infty \psi_{\alpha/2,0}(t,s)S(s)xds, \qquad t > 0, \quad x \in X,$$

where $\psi_{\alpha/2,0}(t,s)$ is the stable Lévy process, see [4, Corollary 14]. Then, by [4, Theorem 3 (vi)]

$$\|S_\alpha(t)\| \le M\int_0^\infty \psi_{\alpha/2,0}(t,s)ds = Mg_{\alpha/2}(t), \quad t > 0,$$

and since $S_\alpha(0) = 0$, $\frac{1}{2} < \frac{\alpha}{2} < 1$ and $(S_\alpha(t))_{t>0}$ is strongly continuous we conclude that $(S_\alpha(t))_{t>0}$ is bounded. Hence, by Theorem 5.3, we obtain a bounded discrete α-resolvent family $\{\mathcal{S}_\alpha(n)\}_{n\in\mathbb{N}_0} \subset \mathcal{B}(X)$.

Our next result imposes a natural and useful condition of compactness on a given family of operators in order to obtain existence and uniqueness of solutions.

Theorem 5.6. *Suppose that A is the generator of a bounded sine family $(S(t))_{t>0}$ on X such that $(\lambda - A)^{-1}$ is a compact operator for some λ large enough. Let $f : \mathbb{N}_0 \times X \to X$ be given and suppose that the condition (F) and the following conditions are satisfied:*

(N) The Nemytskii operator $\mathcal{N}_f$ is continuous in $\ell_h^\infty(\mathbb{N}_0; X)$.

Then, for each $1 < \alpha \le 2$, the problem (15) has an unique solution in $\ell_h^\infty(\mathbb{N}_0; X)$.

Proof. To prove this result we only have to check that the assumption (ii) in Theorem 3.6 is satisfied. Indeed, by hypothesis we have that $(\lambda^\alpha - A)^{-1}$ is compact for all $\lambda^\alpha \in \rho(A)$ and all $1 < \alpha \le 2$. By Example 5.5 we obtain that A is the generator of a bounded α-resolvent family $(S_\alpha(t))_{t>0}$, which is moreover compact by [32, Corollary 2.3]. From Theorem 4.2 (vii) it follows that $\{\mathcal{S}_\alpha(n)\}_{n\in\mathbb{N}_0}$ is compact. Also, for all $a \in \mathbb{N}_0$ and $\sigma > 0$, the set $\{f(k,x) : 0 \le k \le a, \|x\| \le \sigma\}$ is bounded because $\|f(k,x)\| \le \|a\|_\infty(c\|x\| + b) \le \|a\|_\infty(c\sigma + b)$ for all $0 \le k \le a$ and $\|x\| \le \sigma$. Consequently, the set $\{\mathcal{S}_\alpha(n)f(k,x) : 0 \le k \le a, \|x\| \le \sigma\}$ is relatively compact in X for all $n \in \mathbb{N}_0$. □

6. **Examples, applications and final comments.** In this section, we provide several concrete examples and applications of the abstract results developed in the previous sections. Finally we present some related problems with problem (4) for $\alpha = 2$.

Example 6.1. Let $m : [a,b] \to (0,1)$ be a continuous function. Let A be the multiplication operator given by $Af(x) = m(x)f(x)$ defined on $L^2(a,b)$. We known that A is a bounded operator [21, Proposition 4.10, Chapter I]. Since $(\lambda^2 - A)^{-1} = \frac{1}{\lambda^2 - m(x)}$ for λ sufficiently large, we have by Definition 5.1 that A generates a sine family $(S(t))_{t>0}$ on $L^2(a,b)$, given by

$$S(t)f(x) = \frac{1}{2\sqrt{m(x)}}\left(e^{\sqrt{m(x)}t} - e^{-\sqrt{m(x)}t}\right) f(x), \qquad t > 0.$$

Since $0 < m(x) < 1$ we obtain by subordination

$$\begin{aligned}
\mathcal{S}(n)f(x) &= \int_0^\infty p_n(t)S(t)f(x)dt \\
&= \int_0^\infty e^{-t}\frac{t^n}{n!}\frac{1}{2\sqrt{m(x)}}\left(e^{\sqrt{m(x)}t} - e^{-\sqrt{m(x)}t}\right) f(x)dt \\
&= \frac{1}{n!2\sqrt{m(x)}}\left(\int_0^\infty t^n e^{-(1-\sqrt{m(x)})t} f(x)dt - \int_0^\infty t^n e^{-(1+\sqrt{m(x)})t} f(x)dt\right) \\
&= \frac{1}{2\sqrt{m(x)}}\left(\frac{1}{(1-\sqrt{m(x)})^{n+1}} - \frac{1}{(1+\sqrt{m(x)})^{n+1}}\right) f(x),
\end{aligned}$$

for $n \in n \in \mathbb{N}_0$. By Theorem 2.4 and Theorem 5.3, we conclude that the fractional difference equation

$$\Delta^2 u(n) = Au(n+2), \quad n \in \mathbb{N}_0,$$

with initial conditions $u(0) = u_0$ and $u(1) = u_1$, admits the explicit solution

$$\begin{aligned}
u(n) &= (\mathcal{S}(n)(I-A) - 2\mathcal{S}(n-1))u_0 + \mathcal{S}(n-1)(I-A)u_1 \\
&= \sqrt{A^{-1}}\left(\frac{I-A}{2}(1-\sqrt{A})^{-(n+1)} - \frac{I-A}{2}(1+\sqrt{A})^{-(n+1)}\right) u_0 \\
&\quad -\sqrt{A^{-1}}\left((1-\sqrt{A})^{-n} - (1+\sqrt{A})^{-n}\right) u_0 \\
&\quad +\frac{1}{2}\sqrt{A^{-1}}\left((1-\sqrt{A})^{-n} - (1+\sqrt{A})^{-n}\right)(I-A)u_1, \quad n \in \mathbb{N}_0.
\end{aligned}$$

Example 6.2. We study the existence of solutions for the problem

$$\begin{cases}
\Delta^\alpha u(n,x) = u_{xx}(n+2,x) + \dfrac{1}{1+n^3}\dfrac{\cos(u(n,x))}{1+\left(\int_0^\pi |u(n,s)|^2 ds\right)^{1/2}}, \; n \in \mathbb{N}_0; \\
u(0,x) = 0; \quad u(1,x) = 0; \\
u(n,0) = 0; \quad u(n,\pi) = 0;
\end{cases} \tag{19}$$

where $0 < x < \pi$ and $1 < \alpha < 2$. We will use Corollary 5.6.

Let $X = L^2[0,\pi]$ and let us define the operator $A = \dfrac{\partial^2}{\partial x^2}$, on the domain

$$D(A) = \{v \in L^2[0,\pi]/v, v' \text{ absolutely continuous}, v'' \in L^2[0,\pi], v(0) = v(\pi) = 0\}.$$

Observe that the operator A can be written as

$$Av = -\sum_{n=1}^\infty n^2(v, z_n)z_n, \quad v \in D(A),$$

where $z_n(s) := \sqrt{2/\pi}\sin ns$, $n = 1, 2, \ldots$, is an orthonormal set of eigenvectors of A.

Note that A is the infinitesimal generator of a sine family $(S(t))_{t\in\mathbb{R}}$ in $L^2[0,\pi]$, given by

$$S(t)v = \sum_{n=1}^{\infty} \frac{\sin nt}{n}(v, z_n)z_n, \quad v \in L^2[0,\pi].$$

The resolvent of A is given by

$$R(\lambda; A)v = \sum_{n=1}^{\infty} \frac{1}{\lambda + n^2}(v, z_n)z_n, \quad v \in L^2[0,\pi], -\lambda \neq k^2, k \in \mathbb{N}.$$

The compactness of $R(\lambda; A)$ follows from the fact that eigenvalues of $R(\lambda; A)$ are $\lambda_n = \frac{1}{\lambda+n^2}$, $n = 1, 2, \ldots$, and thus $\lim_{n\to\infty} \lambda_n = 0$, see for example [31].

Let us consider the weighted space

$$\ell_h^\infty(\mathbb{N}_0; L^2[0,\pi]) = \left\{ \xi : \mathbb{N}_0 \to L^2[0,\pi] / \sup_{n\in\mathbb{N}} \frac{\|\xi(n)\|_{L^2}}{nn!} < \infty \right\},$$

where the function $h(n) = nn!$ is an admissible weight function (see Example 3.3).

For the function $f : \mathbb{N}_0 \times L^2[0,\pi] \to L^2[0,\pi]$, defined by $f(n,v) := \dfrac{1}{1+n^3}\dfrac{\cos(v)}{1+\|v\|}$, we consider the Nemystkii operator $\mathcal{N}_f$ associated to f. That is, $N_f(u) : \mathbb{N}_0 \to L^2[0,\pi]$ is such that $N_f(u)(n) := f(n, u(n))$ for $u : \mathbb{N}_0 \to L^2[0,\pi]$. Then:

(i) $f(n,0) \neq 0$ for all $n \in \mathbb{N}_0$ and there exists $a(n) = \frac{1}{1+n^3}$ in $l^1(\mathbb{N}_0)$ and $c = 1, b = 0$ such that $\|f(n,v)\| \leq a(n)(c\|v\| + b)$, for all $n \in \mathbb{N}_0$ and $v \in L^2[0,\pi]$.

(ii) Is clear from the definition.

Consequently, by Theorem 5.6, we conclude that the problem (19) has an unique solution $u \in \ell_h^\infty(\mathbb{N}_0)$, that is, u satisfies

$$\sup_{n\in\mathbb{N}_0} \frac{\|u(n)\|_{L^2}}{nn!} = \sup_{n\in\mathbb{N}_0} \frac{1}{nn!}\left(\int_0^\pi |u(n,x)|^2 dx\right)^{1/2} < \infty,$$

Therefore, there exist a constant $K > 0$ such that

$$\int_0^\pi |u(n,x)|^2 dx < K(nn!)^2, \quad n \in \mathbb{N}.$$

Final comments. In some circumstances, the equation (4) for $\alpha = 2$ may have a different format on the right hand side. For instance, the problem

$$\begin{cases} \Delta^2 u(n) &= Bu(n+1) + g(n, u(n)), \quad n \in \mathbb{N}_0; \\ u(0) &= u_0; \\ u(1) &= u_1. \end{cases} \tag{20}$$

where B is a linear operator defined on a Banach space X. In such cases, and under mild conditions, we can still handle this problem with our theory. That is the content of the following two results.

Proposition 6.3. Let B be a linear operator defined on a Banach space X, such that $-2 \in \rho(B)$. Then, (20) is equivalent to the problem

$$\begin{cases} \Delta^2 u(n) &= Tu(n+2) + Tu(n) + (I-T)g(n, u(n)), \quad n \in \mathbb{N}_0; \\ u(0) &= u_0; \\ u(1) &= u_1. \end{cases} \tag{21}$$

where $T = I - 2(2+B)^{-1}$.

Proof. From the definition

$$\Delta^2 u(n) = u(n+2) - 2u(n+1) + u(n),$$

we obtain

$$u(n+1) = \frac{1}{2}(u(n+2) - \Delta^2 u(n) + u(n)).$$

On the other hand, by (20) we have

$$u(n+2) - 2u(n+1) + u(n) = Bu(n+1) + g(n, u(n))$$

that is, for $-2 \in \rho(B)$ we have

$$u(n+1) = (2+B)^{-1}u(n+2) + (2+B)^{-1}u(n) - (2+B)^{-1}g(n, u(n)).$$

By identifying both expressions for $u(n+1)$, we achieve

$$(2+B)^{-1}u(n+2) + (2+B)^{-1}u(n) - (2+B)^{-1}g(n, u(n)) = \frac{1}{2}(u(n+2) - \Delta^2 u(n) + u(n)),$$

and therefore

$$\Delta^2 u(n) = (I - 2(2+B)^{-1})u(n+2) + (I - 2(2+B)^{-1})u(n) + 2(2+B)^{-1}g(n, u(n)).$$

So, assuming $-2 \in \rho(B)$, the original problem (20) is equivalent to the problem (21), with $T = I - 2(2+B)^{-1}$. □

Observe that the operator T in the above proposition is bounded whenever B is a closed linear operator and $-2 \in \rho(B)$. A second case of interest is the following.

Proposition 6.4. Let B be a linear operator defined on a Banach space X, such that $1 \in \rho(B)$. Then, the problem

$$\begin{cases} \Delta^2 u(n) &= Bu(n) + g(n+1, u(n+1)), \quad n \in \mathbb{N}_0; \\ u(0) &= u_0; \\ u(1) &= u_1. \end{cases} \tag{22}$$

is equivalent to the problem

$$\begin{cases} \Delta^2 u(n) &= Tu(n+2) - 2Tu(n+1) + (I-T)g(n+1, u(n+1)), \quad n \in \mathbb{N}_0; \\ u(0) &= u_0; \\ u(1) &= u_1. \end{cases} \tag{23}$$

where $T = I - (I-B)^{-1}$.

Proof. From the definition

$$\Delta^2 u(n) = u(n+2) - 2u(n+1) + u(n),$$

we obtain

$$u(n) = \Delta^2 u(n) - u(n+2) + 2u(n+1).$$

On the other hand, by (22) we have

$$u(n+2) - 2u(n+1) + u(n) = Bu(n) + g(n+1, u(n+1))$$

that is, for $1 \in \rho(B)$ we have

$$u(n) = -(I-B)^{-1}u(n+2) + 2(I-B)^{-1}u(n+1) + (I-B)^{-1}g(n+1, u(n+1)).$$

By identifying both expressions for $u(n)$, we achieve

$$(I-B)^{-1}\left(-u(n+2) + 2u(n+1) + g(n+1, u(n+1))\right) = \Delta^2 u(n) - u(n+2) + 2u(n+1),$$

and therefore

$$\Delta^2 u(n) = (I-(I-B)^{-1})u(n+2)-2(I-(I-B)^{-1})u(n+1)+(I-B)^{-1}g(n+1,u(n+1)).$$

So, assuming $1 \in \rho(B)$, the problem (22) is equivalent to the problem (23), with $T = I - (I-B)^{-1}$. □

For instance, let B be a linear operator defined on a Banach space X, and γ a positive constant. We study the existence of solutions of the problem

$$\begin{cases} \Delta^2 u(n,x) = (B+2\gamma)u(n+1,x), & n \in \mathbb{N}_0, x \in [a,b]; \\ u(0,x) = 0; \quad u(1,x) = 0, \\ u(n,a) = 0; \quad u(n,b) = 0. \end{cases} \tag{24}$$

By Proposition 6.3 the solution of (24) coincides with the solution of the problem

$$\begin{cases} \Delta^2 u(n,x) = Tu(n+2,x) + Tu(n,x), & n \in \mathbb{N}_0, x \in [a,b]; \\ u(0,x) = 0; \quad u(1,x) = 0, \\ u(n,a) = 0; \quad u(n,b) = 0. \end{cases} \tag{25}$$

where $T = I - 2(2(1+\gamma)+B)^{-1}$, provided that $2+2\gamma \in \rho(-B)$.

As an example of application to Theorem 3.4 with $\alpha = 2$, let us consider $X = L^2(\pi, 2\pi)$ and define

$$Bf(x) = 2\Big(\frac{1}{1+x} - (1+\gamma)\Big)f(x), \quad x \in [\pi, 2\pi].$$

Note that B is bounded. A computation shows that $Tf(x) = -xf(x)$ and therefore generates the sine family

$$S(t)f(x) = \frac{\sin(\sqrt{x}t)}{\sqrt{x}}f(x), \quad x \in [\pi, 2\pi]$$

It follows that $\|T\| \leq 2\pi$ and $\|S\|_\infty \leq \sqrt{\pi}$.

Let h the admissible weight function defined by $h(n) = nn!$, for which we have

$$\sup_{n\in\mathbb{N}_0} \frac{1}{h(n)} \sum_{k=0}^{n-2} h(k) = \frac{1}{18}$$

since $\frac{1}{h(n)} \sum_{k=0}^{n-2} h(k)$ is a decreasing sequence for $n \geq 3$. Let us consider the function $f : \mathbb{N}_0 \times X \to X$ defined by $f(n,\xi) = \frac{1}{n^2}T\xi + \frac{1}{1+n^2}$. Then the function f verifies:

(F) $f(n,0) \neq 0$ for all $n \in \mathbb{N}_0$ and there exists $a(n) = \frac{1}{n^2}$ in $\ell^1(\mathbb{N}_0)$ and $c = \|T\|, b = 1$ such that $\|f(n,\xi)\| \leq a(n)(c\|\xi\| + b)$ for all $n \in \mathbb{N}_0$ and $\xi \in X$.

(L) There exists $L := \|T\|$ such that

$$\|f(n,x) - f(n,y)\| \leq \|T\|\|x-y\|,$$

for all $x, y \in X$. Moreover,

$$\|T\|\|S\|_\infty \frac{1}{18} < \frac{2\pi\sqrt{\pi}}{18} < 1.$$

Therefore, by Theorem 3.4 we conclude that the problem (25) has an unique solution or, equivalently, the problem (24) has an unique solution $u \in \ell^\infty_h(\mathbb{N}_0; X)$.

REFERENCES

[1] L. Abadias, M. De León and J. L. Torrea. *Non-local fractional derivatives. Discrete and continuous.* J. Math. Anal. Appl. 449 (1) (2017), 734–755.

[2] L. Abadias and C. Lizama. *Almost automorphic mild solutions to fractional partial difference-differential equations.* Appl. Anal., 95 (6) (2016), 1347-1369.

[3] L. Abadias, C. Lizama, P. J. Miana and M. P. Velasco. *Cesàro sums and algebra homomorphisms of bounded operators.* Israel J. Math. 216 (1) (2016), 471–505.

[4] L. Abadias and P. J. Miana. *A subordination principle on Wright functions and regularized resolvent families.* J. Funct. Spaces, Art. ID 158145, (2015), 9 pp.

[5] T. Abdeljawad and F.M. Atici. *On the definitions of nabla fractional operators.* Abstr. Appl. Anal., volume 2012, (2012), 1-13. doi:10.1155/2012/406757.

[6] G. Akrivis, B. Li and C. Lubich. *Combining maximal regularity and energy estimates for the discretizations of quasilinear parabolic equations.* Math. Comp. 86 (2017), 1527–1552.

[7] W. Arendt, C. Batty, M. Hieber and F. Neubrander. Vector-valued Laplace Transforms and Cauchy Problems. Monographs in Mathematics. vol. 96. Birkhäuser, Basel, 2001.

[8] F. M. Atici and P. W. Eloe. *A transform method in discrete fractional calculus.* Int. J. Difference Equ. 2 (2) (2007), 165–176.

[9] F. M. Atici and P. W. Eloe. *Discrete fractional calculus with the nabla operator.* Electr. J. Qual. Th. Diff. Equ., 3 (2009), 1-12.

[10] F.M. Atici and S. Sengül. *Modeling with fractional difference equations.* J. Math. Anal. Appl., 369 (2010), 1-9.

[11] F.M. Atici and P.W. Eloe. *Linear systems of fractional nabla difference equations.* Rocky Mountain J. Math. 41 (2) (2011), 353–370.

[12] Y. Bai, D. Baleanu and G.C. Wu. *Existence and discrete approximation for optimization problems governed by fractional differential equations.* Commun. Nonlinear Sci. Numer. Simul. 59 (2018), 338—348.

[13] D. Baleanu, K. Diethelm, E. Scalas and J.J. Trujillo, *Fractional Calculus: Models and Numerical Methods.* World Scientific, 2012.

[14] E. Bazhlekova. *Fractional Evolution Equations in Banach Spaces.* Ph.D. Thesis, Eindhoven University of Technology, 2001.

[15] J. Cermák, T. Kisela, and L. Nechvátal.*Stability and asymptotic properties of a linear fractional difference equation.* Adv. Difference Equ. 122 (2012), 1-14.

[16] J. Cermák, T. Kisela, and L. Nechvátal. *Stability regions for linear fractional differential systems and their discretizations.* Appl. Math. Comput. 219 (12) (2013), 7012–7022.

[17] E. Cuesta, C. Lubich and C. Palencia. *Convolution quadrature time discretization of fractional diffusion-wave equations.* Math. Comp. 75 (254) (2006), 673–696.

[18] E. Cuesta and C. Palencia. *A numerical method for an integro-differential equation with memory in Banach spaces: Qualitative properties.* SIAM J. Numer. Anal., 41 (3) (2003), 1232–1241.

[19] E. Cuesta. *Asymptotic behaviour of the solutions of fractional integro-differential equations and some time discretizations.* Discrete Contin. Dyn. Syst. 2007, Dynamical Systems and Differential Equations. Proceedings of the 6th AIMS International Conference, suppl., 277–285.

[20] I.K. Dassios, D.I. Baleanu and G.I. Kalogeropoulos. *On non-homogeneous singular systems of fractional nabla difference equations.* Appl. Math. Comput. 227 (2014), 112–131.

[21] K.J. Engel and R. Nagel. One-parameter semigroups for linear evolution equations. Graduate Texts in Mathematics, 194. Springer-Verlag, New York, 2000.

[22] T. Kemmochi and N. Saito. *Discrete maximal regularity and the finite element method for parabolic equations.* Num. Math. 138 (4) (2018), 905–937.

[23] V. Keyantuo, C. Lizama and M. Warma. *Spectral criteria for solvability of boundary value problems and positivity of solutions of time-fractional differential equations.* Abstr. Appl. Anal., vol. 2013, (2013) 1-11. doi:10.1155/2013/614328.

[24] C. Lizama. *The Poisson distribution, abstract fractional difference equations, and stability.* Proc. Amer. Math. Soc., 145 (9) (2017), 3809–3827.

[25] C. Lizama. *l_p-maximal regularity for fractional difference equations on UMD spaces.* Math. Nach., 288 (17/18) (2015), 2079-2092.

[26] C. Lizama and M. P. Velasco. *Weighted bounded solutions for a class of nonlinear fractional equations.* Fract. Calc. Appl. Anal. 19 (4) (2016), 1010–1030.

[27] C. Lizama and M. Murillo-Arcila. *ℓ_p-maximal regularity for a class of fractional difference equations on UMD spaces: The case $1 < \alpha \leq 2$.* Banach J. Math. Anal. 11 (1) (2017), 188–206.

[28] C. Lizama and M. Murillo-Arcila. *Maximal regularity in ℓ_p spaces for discrete time fractional shifted equations.* J. Differential Equations. 263 (2017), 3175–3196.

[29] K. S. Miller and B. Ross. *Fractional difference calculus.* In: Univalent functions, fractional calculus, and their applications (Köriyama, 1988), Horwood, Chichester, (1989), 139–152.

[30] A. M. Sinclair. Continuous Semigroups in Banach Algebras, London Mathematical Society, Lecture Notes Series 63, Cambridge University Press, New York, 1982.

[31] C.C. Travis and G.F. Webb. *Compactness, regularity, and uniform continuity properties of strongly continuous cosine families.* Houston Journal of Mathematics 3 (4) (1977), 555-567.

[32] L.W. Weis. *A generalization of the Vidav-Jorgens perturbation theorem for semigroups and its application to Transport Theory.* J. Math. Anal. Appl. 129 (1988), 6-23.

[33] G.C. Wu, D. Baleanu and L.L. Huang. *Novel Mittag-Leffler stability of linear fractional delay difference equations with impulse.* Appl. Math. Lett. 82 (2018), 71—78.

[34] G.C. Wu, D. Baleanu and H.P. Xie. *Riesz Riemann–Liouville difference on discrete domains.* Chaos 26 (8) (2016) 084308, 5 pp.

[35] G.C. Wu and D. Baleanu. *Discrete chaos in fractional delayed logistic maps.* Nonlinear Dynam. 80 (4) (2016), 1697–1703.

[36] G.C. Wu and D. Baleanu. *Stability analysis of impulsive fractional difference equations.* Fract. Calc. Appl. Anal. 21 (2) (2018), 354–375.

[37] A. Zygmund. Trigonometric Series. 2nd ed. Vols. I, II, Cambridge University Press, New York, 1959.

Received xxxx 20xx; revised xxxx 20xx.

E-mail address: labadias@unizar.es
E-mail address: carlos.lizama@usach.cl
E-mail address: pjmiana@unizar.es
E-mail address: pvelasco@etsist.upm.es