C-SEMIGROUPS, SUBORDINATION PRINCIPLE AND THE LÉVY α–STABLE DISTRIBUTION ON DISCRETE TIME

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Abstract. In this paper, we introduce the notion of Lévy α-stable distribution within the discrete setting. Using this notion, a subordination principle is proved, which relates a sequence of solution operators - given by a discrete C-semigroup - for the abstract Cauchy problem of first order in discrete-time, with a sequence of solution operators for the abstract Cauchy problem of fractional order $0 < \alpha < 1$ in discrete-time. As an application, we establish the explicit solution of the abstract Cauchy problem in discrete-time that involves the Hilfer fractional difference operator and prove that, in some cases, such solution converges to zero. Our findings gives new insights on the theory, provide original concepts and extend as well as improve recent results of relevant references on the subject.

1. Introduction

The theory of discrete fractional calculus is a new and challenging topic of research that arises in the past three decades, due to their interplay with new analytical concepts as well as numerical methods [12]. Nowadays, the theory is being developed in two main areas of discussion: scalar-valued and operator-valued setting. Whereas the scalar-valued setting is relatively older [5, 7, 10, 12, 13, 15, 24, 28], the vector-valued theory began to be discussed only recently in the reference [22] in which the author analyzed the abstract Cauchy problem in discrete-time

$$\Delta^\alpha u(n) = Au(n+1), \quad u(0) = u_0, \quad n \in \mathbb{N}, \tag{1.1}$$

where A is a closed linear operator defined in a Banach space X and $0 < \alpha \leq 1$. We observe that, in concrete examples, the operator A typically is the negative Laplacian in $X = L^2(\Omega)$, or the elasticity operator, the Stokes operator, or the biharmonic operator, etc. equipped with suitable boundary conditions.

The fractional difference operator Δ^α (Riemann-Liouville like) is defined by means of finite convolution, i.e.:

$$\Delta^\alpha u(n) = (k^{1-\alpha} * u)(n+1) - (k^{1-\alpha} * u)(n), \quad 0 < \alpha \leq 1, \quad n \in \mathbb{N}_0.$$

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where $k^\alpha(n)$ is a sequence initially defined as

$$k^\alpha(n) = \frac{\Gamma(n+\alpha)}{\Gamma(\alpha)\Gamma(n+1)}, \qquad n \in \mathbb{N}_0.$$

From a numerical point of view, the fractional difference operator Δ^α amounts to a convolution quadrature generated by the kernel $z^{1-\alpha}(1-z)^\alpha$. This observation was revealed in the reference [19] by Jin, Li and Zhou, where these authors studied discrete maximal regularity of time-stepping schemes for fractional evolution equations.

One of the striking facts of the fractional difference operator Δ^α is their close relation with the classical Riemann-Liouville fractional operator D_t^α defined as

$$D_t^\alpha f(t) = \int_0^\infty g_{1-\alpha}(t-s) f(s) ds, \quad t \geq 0$$

where

$$g_\alpha(t) = \frac{t^{\alpha-1}}{\Gamma(\alpha)}, \quad t > 0, \quad \alpha > 0.$$

This relationship is given by the following identity

$$\mathcal{P}(D_t^\alpha f)(n+1) = \Delta^\alpha \mathcal{P}(f)(n), \qquad n \in \mathbb{N}_0, \tag{1.2}$$

where $\mathcal{P}$ denotes the Poisson transform, defined by

$$\mathcal{P}(f)(n) := \int_0^\infty e^{-t} \frac{t^n}{n!} f(t) dt, \ n \in \mathbb{N}_0.$$

The Poisson transform was defined in [22] and the property (1.2) was proved in the same reference. The Poisson transform has become one of the main objects that links the discrete-time fractional world with the continuous one. Some of their main properties appeared only recently studied in [3, Section 4].

Because of the identity (1.2), the theory for the abstract Cauchy problem of fractional order in discrete-time (1.1) began to be investigated. Such research, parallels the continuous case and uses methods of operator theory [3, 17, 22]. One of the main tools is the concept of α-resolvent sequences of operators, introduced in [23] and used in subsequent works by several authors, see e.g. [2, 17]. We recall that an α-resolvent sequence $\{S_\alpha(n)\}_{n\in\mathbb{N}_0} \subset \mathcal{B}(X)$ is essentially defined by two conditions:

(*i*) $S_\alpha(n)x \in D(A)$ for all $x \in X$ and $S_\alpha(n)Ax = AS_\alpha(n)x$ for each $n \in \mathbb{N}_0$ and $x \in D(A)$;

(*ii*) $S_\alpha(n)x = k^\alpha(n)x + A(k^\alpha * S_\alpha)(n)x$ for all $n \in \mathbb{N}_0$ and each $x \in X$.

One of the main properties of α-resolvent sequences is that $1 \in \rho(A)$, the resolvent set of A, and that there exists a scalar sequence $\{\beta_{\alpha,n}(j)\}_{n,j\in\mathbb{N}}$ such that

$$S_\alpha(n)x = \sum_{j=1}^{n} \beta_{\alpha,n}(j)(I-A)^{-(j+1)}x, \quad x \in X, \quad n \in \mathbb{N}, \tag{1.3}$$

see [2, Theorem 3.2]. It can be proved that $u : \mathbb{N}_0 \to [D(A)]$ verifies (1.1) if and only if u satisfies $u_0 \in D(A)$ and

$$u(n) = S_\alpha(n)(I - A)u_0, \quad n \in \mathbb{N},$$

see [17, Theorem 4.1]. This result is very important for the theory, because subordinates the solution of the problem (1.1) in case $0 < \alpha < 1$ to the solution of the problem (1.1) in the simple case $\alpha = 1$, namely

$$\mathcal{T}(n) := (I - A)^{-(n+1)}, \quad n \in \mathbb{N}_0.$$

However, a closed and precise description of the scalar sequence $\{\beta_{\alpha,n}(j)\}_{n,j\in\mathbb{N}}$, was not given in such papers and remains as an open problem.

The main purpose of this paper is to solve this open problem and, in passing, clarify the role of the sequence of operators $\{\mathcal{T}(n)\}_{n\in\mathbb{N}_0}$ that appears in the formula (1.3). In fact, we will see (Section 4 below) that $\{\mathcal{T}(n)\}_{n\in\mathbb{N}_0}$ constitutes a *discrete C-semigroup.* This concept is defined here by the first time, but it is the analogous to the notion of C-semigroup introduced by De Laubenfels [8] in the nineties. The surprising fact that we find, when we compare the solutions of the first order abstract Cauchy problem in discrete time (i.e. (1.1) in case $\alpha = 1$) with those of the first order abstract Cauchy problem in continuous time, namely

$$u'(t) = Au(t), \quad u(0) = u_0, \quad t \geq 0,$$

is that whereas the last is well-posed if and only if A generates an strongly continuous semigroup of operators, the former is well-posed if and only if A generates a discrete C-semigroup, where $C := (I - A)^{-1}$. This important observation allows to interpret the formula (1.3) as a sort of subordination principle, concept that already exists in the literature but in continuous time. We remember that the notion of subordination was introduced by Prüss [27] for the theory of integral equations and then used by Bazhlekova [6] in the theory of abstract fractional evolution equations. However, to best of our knowledge, until now a subordination principle on discrete time has not been developed.

In order to state our main result, Theorem A below, and answer the stated open problem, we need to introduce a new notion: The *Lévy α–stable distribution on discrete time.* This is a two parameter sequence of numbers, introduced in Section 3 below, and that can be defined as

$$\ell_\alpha(n,j) = \sum_{i=0}^{j} \binom{j}{i} (-1)^i k^{-\alpha i}(n), \quad 0 < \alpha \leq 1, \quad n \in \mathbb{N}_0, \quad j \in \mathbb{N}_0.$$

We will prove that the Lévy α–stable distribution on discrete-time satisfies essential properties: $0 \leq \ell_\alpha(n,j) \leq 1$, $\sum_{i=0}^{\infty} \ell_\alpha(i,j) = 1$ and

$$\sum_{j=0}^{\infty} \ell_\alpha(n,j)(1+\omega)^{-(j+1)} = \mathcal{E}_{\alpha,\alpha}(-\omega, n),$$

where $\mathcal{E}_{\alpha,\beta}(\lambda,n)$ denotes the *Mittag-Leffler sequence* that we will define as

$$\mathcal{E}_{\alpha,\beta}(\lambda,n) := \sum_{j=0}^{\infty} \lambda^j k^{\alpha j+\beta}(n), \quad \lambda \in \mathbb{C}, \quad \alpha,\beta > 0, \quad n \in \mathbb{N}_0.$$

See Section 2 below. With this notion of Lévy α–stable distribution on discrete time, we finally find

$$\beta_{\alpha,n}(j) = \ell_\alpha(n,j), \quad n \in \mathbb{N}_0, \quad j \in \mathbb{N}_0.$$

Nonetheless, our results will be stated in an even more general form: We introduce the concept of (α,ν)-*resolvent sequence* that includes the notion of α-resolvent sequence when $\alpha = \nu$. In such way, our main theorem reads as follows:

Theorem A *Let $0 < \alpha < 1$, $0 < \nu$ be given. Let $\{\mathcal{T}(n)\}_{n\in\mathbb{N}}$ be a discrete C-semigroup generated by A. Then the family*

$$S_{\alpha,\nu}(n)x = \sum_{j=0}^{\infty} (k^{\nu-\alpha} * \ell_\alpha(\cdot,j))(n)\mathcal{T}(j)x, \quad n \in \mathbb{N}_0,$$

is a discrete (α,ν)-resolvent sequence generated by A.

This result extends and improves [2, Theorem 3.2] and [3, Theorem 2.3].

In order to show how our results apply in other contexts, we introduce a Hilfer fractional difference operator $\Delta^{\alpha,\beta}$ of order $\alpha > 0$ and type $0 \le \beta \le 1$. See Section 5. It should be noted that the first form of the Hilfer fractional difference operator appears to be defined in a more general setting and very recently by Haider et.al. [16], where it is denoted by $\Delta_a^{\mu,\nu}$. After some preliminares about the main properties of $\Delta^{\alpha,\beta}$, and after establishing their relationship with the operators Δ^α and $\Delta_a^{\mu,\nu}$, we first prove an extension of the remarkable identity (1.2) that reads as follows:

$$\mathcal{P}({}_H D^{\alpha,\beta} u)(n+1) = \Delta^{\alpha,\beta}\mathcal{P}(u)(n), \quad n \in \mathbb{N}_0,$$

where ${}_H D^{\alpha,\beta}$ denotes the Hilfer fractional derivative in continuous time. See Theorem 5.9 below. Finally, we find the following interesting result that take into account most of the findings in this paper and reveals a new and surprising asymptotic behavior of the solutions of certain fractional difference equations.

Corollary *Let $0 < \alpha < 1$ and $0 < \beta < 1$ be given. Assume that A generates a discrete C-semigroup $\{\mathcal{T}(n)\}_{n\in\mathbb{N}_0}$ such that $\|\mathcal{T}(n)\| \le M(1+\omega)^{-n}$ for some $M,\omega > 0$. Then the fractional difference equation*

$$\begin{aligned} \Delta^{\alpha,\beta}u(n) &= A\left[u(n+1) - k^{\beta(1-\alpha)}(n+1)u_0\right], \quad n \in \mathbb{N}_0, \\ u(0) &= u_0 \in D(A), \end{aligned}$$

admits the unique solution

$$u(n) = \sum_{j=0}^{\infty} (k^{\beta(1-\alpha)} * \ell_\alpha(\cdot,j))(n)\mathcal{T}(j)C^{-1}u_0, \quad n \in \mathbb{N}_0.$$

Moreover, $u(n) \to 0$ as $n \to \infty$.

It is worth pointing out that our technique is original. For instance, in contrast to the continuous case, the properties of the Lévy α-stable distribution on discrete time should be proved using different methods. As an example, the non-negativity is obtained from its own definition while in the continuous case it is necessary to use the concept of completely monotone function. On the other hand, handling with convergences of the series involved require a more delicate analysis that heavily depends on the kernel k^α and their very special properties.

2. Preliminaries

In this section we start with a general description of the notation and basic results which will be fundamental to understand our theory.

Let X be a complex Banach space equipped with the norm $\|\cdot\|$ and $\mathcal{B}(X)$ denotes the Banach space of all bounded operators defined on X. For a real number a, we denote $\mathbb{N}_a := \{a, a+1, a+2, ...\}$ and when $a = 1$ we write $\mathbb{N}$. The vector space of all vector-valued sequences $f : \mathbb{N}_a \to X$ will be denoted by $s(\mathbb{N}_a; X)$. The forward Euler operator $\Delta_a : s(\mathbb{N}_a; X) \to s(\mathbb{N}_a; X)$ is defined by

$$\Delta_a f(t) := f(t+1) - f(t), \quad t \in \mathbb{N}_a.$$

For $m \in \mathbb{N}_2$, we define recursively $\Delta_a^m : s(\mathbb{N}_a; X) \to s(\mathbb{N}_a; X)$ by

$$\Delta_a^m := \Delta_a^{m-1} \circ \Delta_a,$$

and is called the m-th order forward difference operator.

We recall that the discrete convolution $*$ of two sequences $f, g \in s(\mathbb{N}_0; X)$ is defined by

$$(f * g)(n) := \sum_{j=0}^{n} f(n-j)g(j),$$

and the $\mathcal{Z}$-transform of a vector-valued sequence $f \in s(\mathbb{N}_0; X)$, is defined by

$$\widetilde{f}(z) := \sum_{j=0}^{\infty} f(j) z^{-j},$$

where z is a complex number. Let Γ be a circle centered at the origin of the z-plane that encloses all poles of $\widetilde{f}(z)z^{n-1}$. We recall that the inverse $\mathcal{Z}$-transform of $\widetilde{f}(z)$ is defined by

$$f^\vee(n) = \frac{1}{2\pi i} \int_\Gamma \widetilde{f}(z) z^{n-1}\, dz. \tag{2.1}$$

For more details about $\mathcal{Z}$-transforms and its inverse, see [9, Chapter 6]. For $f \in s(\mathbb{N}_0; X)$ and $m \in \mathbb{N}$ the m–order difference operator Δ^m is defined by

$$\Delta^m f(n) := \sum_{j=0}^{m} (-1)^{m-j} \binom{m}{j} u(n+j), \quad n \in \mathbb{N}_0. \tag{2.2}$$

We also define $\Delta^0 \equiv I$ where I is the identity operator. We define the translation (by $a \in \mathbb{R}$) operator $\tau_a : s(\mathbb{N}_a; X) \to s(\mathbb{N}_0; X)$ by

$$\tau_a g(n) := g(a+n), \quad n \in \mathbb{N}_0.$$

Note that $\tau_a^{-1} = \tau_{-a}$ and $\tau_{a+b} = \tau_a \circ \tau_b = \tau_b \circ \tau_a$. Moreover, $\Delta_a^m \circ \tau_a^{-1} = \tau_a^{-1} \circ \Delta_0^m$. In other words, the following diagram is commutative

$$\begin{array}{ccc} s(\mathbb{N}_a; X) & \overset{\Delta_a^m}{\longrightarrow} & s(\mathbb{N}_a; X) \\ \downarrow \tau_a & & \downarrow \tau_a \\ s(\mathbb{N}_0; X) & \underset{\Delta^m}{\longrightarrow} & s(\mathbb{N}_0; X). \end{array} \tag{2.3}$$

In particular, we have

$$\Delta_a^m f(a+n) = \sum_{j=0}^{m} \binom{m}{j} (-1)^{m-j} f(a+n+j), \quad n \in \mathbb{N}_0. \tag{2.4}$$

For an arbitrary $\alpha \in \mathbb{C} \setminus \{0\}$, we consider the scalar sequence $\{k^\alpha(n)\}_{n \in \mathbb{N}_0}$ defined by

$$k^\alpha(n) := \begin{cases} \dfrac{\alpha(\alpha+1)\cdots(\alpha+n-1)}{n!}, & n \in \mathbb{N} \\ 1, & n = 0. \end{cases} \tag{2.5}$$

Moreover, we define $k^0(n) := \delta_0(n)$ the Kronecker delta. The sequence k^α was introduced in the context of differences of fractional order by Lizama in the reference [22] and a survey of their main properties is given in [11]. From the definition it is clear that $k^\alpha(1) = \alpha$ and, if $\alpha \in \mathbb{Z}^-$, then $k^\alpha(n) = 0$ for all $n \geq |\alpha| + 1$. Obviously, $k^\alpha(n) \geq 0$ for all $\alpha > 0$ and $n \in \mathbb{N}_0$.

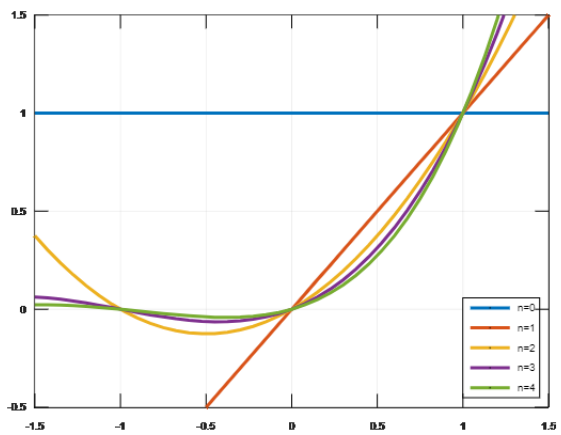

FIGURE 1. The graph of $\alpha \to k^\alpha(n)$ for $\alpha \in \mathbb{R}$ and $n = 0, 1, 2, 3, 4$.

Sometimes, to perform computations with the sequence k^α, we use an equivalent formulation of (2.5), namely

$$k^\alpha(n) = \begin{cases} \dfrac{\Gamma(n+\alpha)}{\Gamma(\alpha)\Gamma(n+1)}, & \alpha \in \mathbb{C}\setminus\mathbb{Z}^- \\ \delta_0(n), & \alpha = 0, \end{cases} \tag{2.6}$$

where $\Gamma(\cdot)$ denotes the Gamma function. The sequence k^α satisfies the following group property

$$(k^\alpha * k^\beta)(n) = k^{\alpha+\beta}(n), \tag{2.7}$$

for all $\alpha, \beta \in \mathbb{C}$, see [11]. On the other hand, we have the identity

$$\Delta^m k^\alpha(n) = \prod_{j=1}^{m} \left(\frac{\alpha - j}{n + j}\right) k^\alpha(n), \qquad m, n \in \mathbb{N}_0, \quad \alpha > 0, \tag{2.8}$$

see [11] for a proof. Furthermore, the following estimate holds: for $0 < \alpha < 1$,

$$\frac{1}{\Gamma(\alpha)(n+1)^{1-\alpha}} < k^\alpha(n) < \frac{1}{\Gamma(\alpha)n^{1-\alpha}}, \tag{2.9}$$

see [4, Section 2]. In addition, the sequence $n \mapsto k^\alpha(n)$ is increasing when $\alpha > 1$, decreasing for $0 < \alpha < 1$ and k^α is of constant sign for n large enough, whenever $\alpha < 0$, see [30, Vol. I, p.77(1.18)] or [11]. We recall that, for $\alpha \in \mathbb{C}\setminus\mathbb{Z}^-$, the sequence k^α satisfies the generation formula

$$\sum_{n=0}^{\infty} k^\alpha(n) w^n = (1-w)^{-\alpha}, \quad |w| < 1, \tag{2.10}$$

see [30, Vol. I, p.76(1.9)] and [11, Proposition 3.1]. Although in the mentioned papers the authors exclude the non-positive integers, by the representation (2.5) and [14, Formula 1.111], the identity (2.10) is valid for all $\alpha \in \mathbb{C}$. Replacing $w = 1/z$, with $z \in \mathbb{C}\setminus\{0\}$, in (2.10), we find that the $\mathcal{Z}$-transform of k^α is given by the formula

$$\widetilde{k^\alpha}(z) = \left(\frac{z}{z-1}\right)^\alpha. \tag{2.11}$$

Note that for $\alpha \in \{0\} \cup \mathbb{Z}^-$ the representation (2.11) is valid for all $z \in \mathbb{C}\setminus\{0\}$, while for all $\alpha \in \mathbb{C}\setminus\{0\}\cup\mathbb{Z}^-$ the expression is valid only for $|z| > 1$.

Additional properties of k^α can be found in the recent reference [11].

The following definition of fractional sum and fractional difference operators (in the sense of Riemann-Liouville and Caputo) was given by Lizama in [22].

Definition 2.1. For $\alpha \geq 0$, the α-th fractional sum of a sequence $f \in s(\mathbb{N}_0; X)$ is defined by means of the formula

$$\Delta^{-\alpha} f(n) := \sum_{j=0}^{n} k^\alpha(n-j) f(j), \qquad n \in \mathbb{N}_0.$$

Where, for $\alpha = 0$, $\Delta^{-\alpha} f(n) = f(n)$.

The following property holds (see [22])

$$\Delta^{-\alpha}\Delta^{-\beta} = \Delta^{-(\alpha+\beta)} = \Delta^{-\beta}\Delta^{-\alpha}, \qquad \alpha, \beta > 0. \tag{2.12}$$

It should be noted that in the special case of $f(j) = T^j$, where $T \in \mathcal{B}(X)$, i.e. $f(j)$ is a discrete semigroup and that the notion of fractional sum of order $\alpha > 0$ corresponds to the concept of Cesáro sums [4, Section 3]. Characterizations of sequences of operators which are Cesáro sums appear in [4, Theorem 3.3] as well as their connection with certain algebra homomorphisms.

In what follows, recall that $\lceil \cdot \rceil$ denotes the ceiling number i.e. $\lceil \alpha \rceil$ is the smallest integer not less than α.

Definition 2.2. Let $f \in s(\mathbb{N}_0; X)$ and $\alpha > 0$ be given. The α-th fractional difference in the sense of Riemann-Liouville of f is defined by

$$_{RL}\Delta^{\alpha} f(n) := \Delta^{m}\Delta^{-(m-\alpha)} f(n), \qquad n \in \mathbb{N}_0, \tag{2.13}$$

where $m - 1 < \alpha \leq m$, $m = \lceil \alpha \rceil$.

The following result was proved in [3, Example 1.3].

Remark 2.3. For $m \in \mathbb{N}$, let $m - 1 < \alpha < m$ and $\alpha < \tau$ be given. Then

$$_{RL}\Delta^{\alpha} k^{\tau}(n) = k^{\tau-\alpha}(n+m), \qquad n \in \mathbb{N}_0. \tag{2.14}$$

The $\mathcal{Z}$-transform of the above defined difference operators are given by

$$\widetilde{\Delta^{-\alpha} f}(z) = \left(\frac{z}{z-1}\right)^{\alpha} \widetilde{f}(z), \quad |z| > 1, \tag{2.15}$$

$$\widetilde{_{RL}\Delta^{\alpha} f}(z) = z\left(\frac{z}{z-1}\right)^{-\alpha} \widetilde{f}(z) - zf(0), \quad |z| > 1, \tag{2.16}$$

where $\alpha \in (0, 1]$, see [25, Propositions 4 and 24].

Definition 2.4. Let $f \in s(\mathbb{N}_0; X)$ and $\alpha > 0$. The Caputo fractional difference of order α is defined by

$$_{C}\Delta^{\alpha} f(n) := \Delta^{-(m-\alpha)}\Delta^{m} f(n), \qquad n \in \mathbb{N}_0,$$

where $m - 1 < \alpha \leq m$, $m = \lceil \alpha \rceil$.

We have the following relation between the Caputo and Riemann-Liouville fractional difference operators of order $0 < \alpha < 1$ and $1 < \alpha < 2$, respectively (see [22, Theorem 2.4.] and [3, Theorem 2.5.]).

Theorem 2.5. Let $f \in s(\mathbb{N}_0; X)$. For each $0 < \alpha < 1$, we have

$$_{C}\Delta^{\alpha} f(n) = {}_{RL}\Delta^{\alpha} f(n) - k^{1-\alpha}(n+1)f(0), \tag{2.17}$$

and for each $1 < \alpha < 2$,

$$_{C}\Delta^{\alpha} f(n) = {}_{RL}\Delta^{\alpha} f(n) - k^{2-\alpha}(n+1)[f(1) - 2f(0)] - k^{2-\alpha}(n+2)f(0). \tag{2.18}$$

Remark 2.6. It is interesting to observe that using (2.14), the identity (2.17) is equivalent to

$$ {}_C\Delta^\alpha f(n) = {}_{RL}\Delta^\alpha f(n) - {}_{RL}\Delta^\alpha k^1(n)f(0) = {}_{RL}\Delta^\alpha\left(f(n) - f(0)\right). $$

Let $\alpha, \beta > 0$ and $\lambda \in \mathbb{C}$. The *Mittag-Leffler sequence* is defined by

$$ \mathcal{E}_{\alpha,\beta}(\lambda, n) := \sum_{j=0}^{\infty} \lambda^j k^{\alpha j+\beta}(n). \tag{2.19} $$

Note that by (2.9) the series in the right hand side of (2.19) converges and can be estimated by $n^{\beta-1}E_{\alpha,\beta}(|\lambda|n^\alpha)$, where $E_{\alpha,\beta}$ denotes the Mittag-Leffler function. See (2.22) below. The $\mathcal{Z}$-transform of the Mittag-Leffler sequence is given by

$$ \widetilde{\mathcal{E}_{\alpha,\beta}}(\lambda, z) = \left(\frac{z}{z-1}\right)^\beta \left(1 - \lambda\left(\frac{z}{z-1}\right)^\alpha\right)^{-1} = \widetilde{k^\beta}(z)(1-\lambda\widetilde{k^\alpha}(z))^{-1}, \tag{2.20} $$

where $|z| > 1$. For more details see [22, 25].

Let $\alpha > 0$ be given and denote

$$ g_\alpha(t) := \begin{cases} \dfrac{t^{\alpha-1}}{\Gamma(\alpha)}, & t > 0 \\ 0, & t < 0. \end{cases} $$

The Riemann-Liouville fractional integral of order $\alpha > 0$ of a absolutely integrable function $u : [0, \infty) \to X$ is given by:

$$ I_t^\alpha u(t) := (g_\alpha * u)(t) := \int_0^t g_\alpha(t-s)u(s)ds. $$

The Riemann-Liouville fractional derivative of order $0 < \alpha < 1$ of a absolutely integrable function u is defined by

$$ {}_{RL}D_t^\alpha u(t) := \frac{d}{dt}(I_t^{1-\alpha}u(t)) := \frac{d}{dt}(g_{1-\alpha} * u)(t). $$

The Hilfer fractional derivative of order $0 < \alpha < 1$ and type $0 \le \beta \le 1$ for an absolutely integrable function u is defined in [18] by

$$ {}_H D_t^{\alpha,\beta} u(t) := I_t^{\beta(1-\alpha)} \frac{d}{dt}(I_t^{(1-\nu)} f(t)) := I_t^{\beta(1-\alpha)} {}_{RL}D_t^\nu u(t), \tag{2.21} $$

where $\nu := \alpha + \beta(1-\alpha)$.

We recall that the classical Mittag-Leffler function is defined as follows

$$ E_{\alpha,\beta}(z) := \sum_{n=0}^{\infty} \frac{z^n}{\Gamma(\alpha n + \beta)}, \qquad \alpha > 0,\ \beta > 0,\ z \in \mathbb{C}. \tag{2.22} $$

One of the more widely used properties of the Mittag-Leffler function is regarding with its Laplace transform:

$$ \int_0^\infty e^{-\lambda t} t^{\beta-1} E_{\alpha,\beta}(\pm\omega t^\alpha)dt = \frac{\lambda^{\alpha-\beta}}{\lambda^\alpha \mp \omega}, \qquad Re(\lambda) > |\omega|^{1/\alpha}, \tag{2.23} $$

see [26, Section 1.2, formula (1.80)]. We recall that the Lévy α-stable distribution (also called Lévy probability density function or stable Lévy process) is defined as follows

$$f_{t,\alpha}(\lambda) = \frac{1}{2\pi i}\int_{\sigma-i\infty}^{\sigma+i\infty} e^{z\lambda - tz^{\alpha}}\, dz, \qquad \sigma > 0,\ t > 0,\ \lambda \geq 0,\ 0 < \alpha < 1, \tag{2.24}$$

where the branch of z^{α} is taken such that $Re(z^{\alpha}) > 0$ for $Re(z) > 0$. This branch is single-valued in the z-plane and cut along the negative real axis, see [29, p.260-262]. For each $n \in \mathbb{N}_0$, the Poisson distribution (with parameter t) is defined by

$$p_n(t) := e^{-t}\frac{t^n}{n!}, \qquad t \geq 0.$$

Given a continuous function $u : [0,\infty) \to X$ the Poisson transform of u was defined in [22] by

$$\mathcal{P}(u)(n) := \int_0^{\infty} p_n(t)u(t)dt,\ n \in \mathbb{N}_0. \tag{2.25}$$

One of the key properties of the Poisson transform is the following relationship between the kernel function g_{α} and the kernel sequence k^{α} :

$$\mathcal{P}(g_{\alpha})(n) = k^{\alpha}(n).$$

Additional properties of the Poisson transform are given in the reference [3, Section 4]. The following result shows a very important relation between the discrete and continuous Riemann-Liouville fractional difference and differential operators, respectively, given through the Poisson transform.

Theorem 2.7. ([22, Theorem 3.5]) Let $u : [0,\infty) \to X$ be locally integrable and bounded. Then we have

$$\mathcal{P}({}_{RL}D_t^{\alpha}u)(n+1) = {}_{RL}\Delta^{\alpha}\mathcal{P}(u)(n), \qquad n \in \mathbb{N}_0.$$

Finally, for a function f defined for $t \geq 0$, we will denote the Laplace transform of $f(t)$ as $\widehat{f}(s)$.

We now introduce the following function:

$$L_{\alpha,\beta}(\lambda,t) := t^{\beta-1}E_{\alpha,\beta}(\lambda t^{\alpha}), \quad t \geq 0, \quad \lambda \in \mathbb{C}, \quad \alpha,\beta > 0. \tag{2.26}$$

We finish this section proving the following result that relates the discrete and continuous Mittag-Leffler functions by means of the Poisson transform.

Theorem 2.8. For all $\lambda \in \mathbb{C}$ we have

$$\mathcal{P}(L_{\alpha,\beta}(\lambda,\cdot))(n) = \mathcal{E}_{\alpha,\beta}(\lambda,n), \quad n \in \mathbb{N}_0.$$

Proof. An easy calculation, using the definitions, shows the following identities

$$\mathcal{P}(L_{\alpha,\beta}(\lambda,\cdot))(n) = \int_0^{\infty} e^{-t}\frac{t^n}{n!}\sum_{j=0}^{\infty}\frac{\lambda^j t^{\alpha j+\beta-1}}{\Gamma(\alpha j+\beta)}dt = \sum_{j=0}^{\infty}\Big[\int_0^{\infty} e^{-t}\frac{t^{n+\alpha j+\beta-1}}{n!\Gamma(\alpha j+\beta)}dt\Big]\lambda^j.$$

Using the identity

$$\int_0^\infty e^{-wt}\frac{t^{\gamma-1}}{\Gamma(\gamma)}dt = \frac{1}{w^{\gamma-1}} \tag{2.27}$$

with $w = 1$ and $\gamma = n + \alpha j + \beta$ produces

$$\sum_{j=0}^{\infty}\Big[\int_0^\infty e^{-t}\frac{t^{n+\alpha j+\beta-1}}{n!\Gamma(\alpha j+\beta)}dt\Big]\lambda^j = \sum_{j=0}^{\infty}\frac{\Gamma(\alpha j+n+\beta)}{n!\Gamma(\alpha j+\beta)}\lambda^j = \sum_{j=0}^{\infty}\lambda^j k^{\alpha j+\beta}(n) = \mathcal{E}_{\alpha,\beta}(\lambda, n).$$

□

3. The discrete Lévy α-stable distribution

In this section, we introduce and investigate the properties of the discrete Lévy α-stable distribution which we propose as an analogous to the continuous Lévy α-stable distribution (2.24). In the context of the study of evolution equations in discrete time by methods of operator theory, and as in the continuous case, this function will be fundamental in order to describe, among others, certain subordination formula, which will be presented in the next section.

Definition 3.1. Let $0 < \alpha \le 1$ be given. For $n \in \mathbb{N}$, the discrete Lévy α-stable distribution is defined by

$$\ell_\alpha(n,j) := \begin{cases} \dfrac{1}{2\pi i}\displaystyle\int_\Gamma z^{n-1}\left(\frac{z^\alpha - (z-1)^\alpha}{z^\alpha}\right)^j dz, & j \in \mathbb{N} \\ \delta_0(n), & j = 0, \end{cases} \tag{3.1}$$

where Γ is a path oriented counterclockwise that encloses all the singularities of the complex variable function $z \to z^{-\alpha j}(z^\alpha - (z-1)^\alpha)^j$.

Denote by $\mathbb{D}(a,r) \subset \mathbb{C}$ the open disk of center $a \in \mathbb{C}$ and radius $r > 0$. The following Lemma will be very useful in what follows.

Lemma 3.2. Let $0 < \alpha \le 1$ be given and $z \in \mathbb{D}(1,1)$. Then $z^\alpha \in \mathbb{D}(1,1)$.

Proof. Write $z = re^{i\theta}$. By hypothesis, $|\theta| < \pi/2$. We first claim that $(\cos\theta)^\alpha < \cos(\alpha\theta)$. In fact, it is enough to prove the claim for $0 < \theta < \pi/2$ (since $\cos x$ is even). Since $\cos x$ is positive and decreasing on the interval $[0, \pi/2]$ and $0 < \alpha\theta < \theta$, we have $\cos(\alpha\theta) > \cos\theta > 0$. Therefore $\ln(\cos\theta) < \ln(\cos\alpha\theta)$. Hence, the condition $0 < \alpha < 1$ implies $\alpha\ln(\cos\theta) < \ln(\cos\alpha\theta)$. This proves the claim. Now, using the claim and the inequality $2^\alpha \le 2$ we obtain $2^\alpha(\cos\theta)^\alpha \le 2\cos(\alpha\theta)$. It shows that if $r < 2\cos\theta$ then $r^\alpha < 2\cos(\alpha\theta)$, or, equivalently, that $|1 - z| < 1$ implies $|1 - z^\alpha| < 1$, proving the lemma. □

Remark 3.3. We observe that the integral on the right hand side of (3.1) contains the analytic function $z \to \dfrac{z^\alpha - (z-1)^\alpha}{z^\alpha} = 1-(1-\dfrac{1}{z})^\alpha$. Suppose $|z|>1$. Then $1-\dfrac{1}{z} \in \mathbb{D}(1,1)$. By the above Lemma, we obtain $(1-\dfrac{1}{z})^\alpha \in \mathbb{D}(1,1)$. We conclude that

$$|1-(1-\frac{1}{z})^\alpha| < 1 \quad \text{for all } |z|>1. \tag{3.2}$$

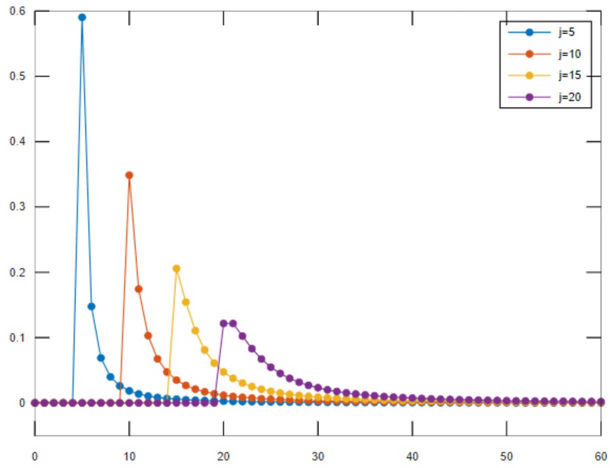

FIGURE 2. The discrete Lévy α-stable distribution for $\alpha = 0.9$ and $0 \le n \le 60$

Next, we present some fundamental properties of the discrete Lévy α-stable distribution. These will be very useful later.

Proposition 3.4. Let $0<\alpha\le 1$, $j \in \mathbb{N}$ and $n \in \mathbb{N}_0$ be given. The following properties hold:

(*i*) $\widetilde{\ell_\alpha}(z,j) = (1-\widetilde{k^{-\alpha}}(z))^j$, for all $|z|>1$.

(*ii*) $\displaystyle\sum_{i=0}^{\infty} \ell_\alpha(i,j) = 1.$

(*iii*) $\ell_\alpha(n,j) = \displaystyle\sum_{i=0}^{j} \binom{j}{i} (-1)^i k^{-\alpha i}(n).$

(*iv*) $0 \le \ell_\alpha(n,j) \le 1.$

(*v*) $(k^\beta * \ell_\alpha(\cdot,j))(n) = \displaystyle\sum_{i=0}^{j} \binom{j}{i} (-1)^i k^{\beta-\alpha i}(n), \quad \beta \in \mathbb{C}.$

(*vi*) $\Delta\, \ell_\alpha(n,\cdot) = (k^{-\alpha} * \ell_\alpha(\cdot,j))(n).$

(*vii*) $\Delta^2 \ell_\alpha(n,\cdot) = (k^{-2\alpha} * \ell_\alpha(\cdot, j))(n)$.

(*viii*) Let $\omega > 0$ be given. Then

$$\sum_{j=0}^{\infty} \ell_\alpha(n,j)(1+\omega)^{-(j+1)} = \mathcal{E}_{\alpha,\alpha}(-\omega, n).$$

(*ix*) Let $\omega > 1$ be given. Then

$$\sum_{i=0}^{\infty} \widetilde{\ell_\alpha}(z,i)\omega^{-i} = \omega \widehat{L}_{\alpha,\alpha}(1-\omega,\cdot)(1-1/z), \quad |z|>1.$$

(*x*) $\mathcal{P}(\widehat{f}_{\cdot,\alpha}(\lambda))(j-1) = \widetilde{\ell_\alpha}(z,j)$ where $|z|>1$ and $\lambda := \left(\dfrac{(z-1)^\alpha}{z^\alpha - (z-1)^\alpha}\right)^{1/\alpha}$.

Proof.

(*i*) Note that, using (2.11) we obtain the identity

$$\ell_\alpha(n,j) = \frac{1}{2\pi i}\int_\Gamma z^{n-1}\left(\frac{z^\alpha - (z-1)^\alpha}{z^\alpha}\right)^j dz = \frac{1}{2\pi i}\int_\Gamma z^{n-1}\left(1 - \widetilde{k^{-\alpha}}(z)\right)^j dz.$$

Then, by the inverse $\mathcal{Z}$-transform, we get the claimed property.

(*ii*) Let $z \in \mathbb{R}$ be given and note that the claimed identity is a particular case of (*i*) by letting $z \to 1^+$ and taking into account that $\widetilde{k^{-\alpha}}(1) = 0$, again by (2.11).

(*iii*) Applying the general binomial theorem (see [14, Formula 1.111]) on (*i*), we get

$$\widetilde{\ell_\alpha}(z,j) = \sum_{i=0}^{j}\binom{j}{i}(-1)^i \widetilde{k^{-\alpha i}}(z),$$

where we have used the group property (2.7) of the sequence kernel k^β to deduce that $\widetilde{k^{-\alpha i}}(z) = [\widetilde{k^{-\alpha}}(z)]^i$. Then the result follows by an application of the inverse $\mathcal{Z}$-transform.

(*iv*) By (*iii*), we have

$$\ell_\alpha(n,1) = k^0(n) - k^{-\alpha}(n) = \frac{\alpha(1-\alpha)(2-\alpha)\dots(n-1-\alpha)}{n!},$$

and, since $0 < \alpha \le 1$, we deduce $\ell_\alpha(n,1) \ge 0$ for all $n \in \mathbb{N}$. Moreover, $\ell_\alpha(0,1) = 0$ by Definition 2.6. On the other hand, observe that

$1-2\alpha \le 1-\alpha,\ 2-2\alpha \le 2-\alpha,\ 3-2\alpha \le 3-\alpha, \dots n-1-2\alpha \le n-1-\alpha.$

Since $i - \alpha > 0$ for all $i = 1, ..., n-1$, then

$$\prod_{i=1}^{n-1}(i-2\alpha) \le \prod_{i=1}^{n-1}(i-\alpha). \tag{3.3}$$

Multiplying by $-2\alpha/n!$ in (3.3) we get

$$-\frac{2\alpha}{n!}\prod_{i=1}^{n-1}(i-2\alpha) \geq -\frac{2\alpha}{n!}\prod_{i=1}^{n-1}(i-\alpha)$$

or

$$\frac{(-2\alpha)(1-2\alpha)\dots(n-1-2\alpha)}{n!} \geq 2\frac{(-\alpha)(1-\alpha)\dots(n-1-\alpha)}{n!}.$$

Equivalently, and by definition of k^α

$$k^{-2\alpha}(n) \geq 2k^{-\alpha}(n).$$

Now, note from (iii) that

$$\ell_\alpha(n,2) = k^0(n) - 2k^{-\alpha}(n) + k^{-2\alpha}(n).$$

From which we deduce

$$\ell_\alpha(n,2) = -2k^{-\alpha}(n) + k^{-2\alpha}(n) \geq -2k^{-\alpha}(n) + 2k^{-\alpha}(n) = 0, \quad n \in \mathbb{N},$$

and $\ell_\alpha(0,2) = 0$ by Definition 2.6. Now, observe that the previous calculations and (the proof of) (i) we obtain

$$\begin{aligned}
\ell_\alpha(n,3) &= \frac{1}{2\pi i}\int_\Gamma z^{n-1}(1-\widetilde{k^{-\alpha}}(z))^3\,dz \\
&= \frac{1}{2\pi i}\int_\Gamma z^{n-1}(1-\widetilde{k^{-\alpha}}(z))^2(1-\widetilde{k^{-\alpha}}(z))\,dz \\
&= \frac{1}{2\pi i}\int_\Gamma z^{n-1}\widetilde{\ell_\alpha}(n,2)\widetilde{\ell_\alpha}(n,1)\,dz \\
&= \sum_{p=0}^{n}\ell_\alpha(n-p,2)\ell_\alpha(p,1) \geq 0, \quad n \in \mathbb{N}_0.
\end{aligned}$$

Now, assume that for $j = m \in \mathbb{N}$ we have

$$\ell_\alpha(n,m) \geq 0, \quad n \in \mathbb{N}.$$

Then, for $j = m+1$ and proceeding as in the case $j = 3$, we obtain

$$\ell_\alpha(n,m+1) = \sum_{p=0}^{n}\ell_\alpha(n-p,m)\ell_\alpha(p,1) \geq 0,$$

for all $n \in \mathbb{N}$. Finally, by (ii) we obtain $\ell_\alpha(n,j) \leq 1$. This proves the claim.

(v) From (iii) and the group property (2.7) of k^α, we have

$$\begin{aligned}
(k^\beta * \ell_\alpha(\cdot,j))(n) &= \sum_{p=0}^{n}k^\beta(n-p)\sum_{i=0}^{j}\binom{j}{i}(-1)^i k^{-\alpha i}(p) \\
&= \sum_{i=0}^{j}\binom{j}{i}(-1)^i(k^\beta * k^{-\alpha i})(n) = \sum_{i=0}^{j}\binom{j}{i}(-1)^i k^{\beta-\alpha i}(n).
\end{aligned}$$

(vi) By (i) we obtain that

$$\begin{aligned}\ell_\alpha(n,j)-\ell_\alpha(n,j+1) &= \frac{1}{2\pi i}\int_\Gamma z^{n-1}\left[(1-\widetilde{k^{-\alpha}}(z))^j-(1-\widetilde{k^{-\alpha}}(z))^{j+1}\right]\,dz\\ &= \frac{1}{2\pi i}\int_\Gamma z^{n-1}\widetilde{k^{-\alpha}}(z)(1-\widetilde{k^{-\alpha}}(z))^j\,dz\\ &= \frac{1}{2\pi i}\int_\Gamma z^{n-1}\widetilde{k^{-\alpha}}(z)\widetilde{\ell_\alpha}(z,j)\,dz.\end{aligned}$$

Applying the inverse $\mathcal{Z}$-transform for the finite convolution, we get the claim.

(vii) By Definition (see (2.2)) we have

$$\Delta^2\ell_\alpha(n,\cdot)=\ell_\alpha(n,j+2)-2\ell_\alpha(n,j+1)+\ell_\alpha(n,j).$$

Following the same steps of the previous proof, we see that

$$\Delta^2\ell_\alpha(n,\cdot)=\frac{1}{2\pi i}\int_\Gamma z^{n-1}\widetilde{k^{-2\alpha}}(z)\widetilde{\ell_\alpha}(z,j)\,dz.$$

The property follows by applying the inverse $\mathcal{Z}$-transform for the finite convolution.

$(viii)$ Using (2.20), one has

$$\widetilde{\mathcal{E}_{\alpha,\alpha}}(-\omega,z)=\frac{z^\alpha}{(z-1)^\alpha+z^\alpha\omega}. \tag{3.4}$$

On the other hand, by Definition 3.1 and taking into account the properties of the $\mathcal{Z}$-transform, we get

$$\begin{aligned}\sum_{j=0}^{\infty}\ell_\alpha(n,j)(1+\omega)^{-(j+1)} &= \frac{1}{2\pi i}\int_\Gamma z^{n-1}\sum_{j=0}^{\infty}\left(\frac{z^\alpha-(z-1)^\alpha}{z^\alpha}\right)^j(1+\omega)^{-(j+1)}dz\\ &= \frac{1}{2\pi i}\int_\Gamma z^{n-1}\frac{z^\alpha}{(z-1)^\alpha+z^\alpha\omega}\,dz,\end{aligned} \tag{3.5}$$

where the geometric series converges because $\omega>0$ and $(1-\frac{1}{z})^\alpha\in\mathbb{D}(1,1)$ for $|z|>1$. See Remark 3.3. Then, by applying the inverse $\mathcal{Z}$-transform, (3.4) and (3.5), we arrive at the conclusion.

(ix) Using geometric series and Remark 3.3 we find, for all $\omega>1$, that

$$\sum_{i=0}^{\infty}\widetilde{\ell_\alpha}(z,i)\omega^{-i}=\sum_{i=0}^{\infty}\left(\frac{z^\alpha-(z-1)^\alpha}{z^\alpha}\right)^i\omega^{-i}=\frac{\omega z^\alpha}{(z-1)^\alpha+z^\alpha(\omega-1)}.$$

On the other hand, by the Laplace transform property (2.23), we have

$$\omega\widehat{L}_{\alpha,\alpha}(-(\omega-1),\cdot)(\lambda)=\frac{\omega}{\lambda^\alpha+(\omega-1)}.$$

Evaluating at $\lambda = 1 - 1/z$, we obtain

$$\omega\widehat{L}_{\alpha,\alpha}(-(\omega-1),\cdot)(1-1/z) = \frac{\omega}{(1-1/z)^{\alpha}+(\omega-1)} = \frac{\omega}{\left(\dfrac{z-1}{z}\right)^{\alpha}+(\omega-1)}$$
$$= \frac{\omega z^{\alpha}}{(z-1)^{\alpha}+z^{\alpha}(\omega-1)}.$$

(x) By (2.24), we have $\widehat{f}_{s,\alpha}(\lambda) = e^{-s\lambda^{\alpha}}$ for all $\operatorname{Re}\lambda > 0$. Then, using the Poisson transform, we get

$$\mathcal{P}(\widehat{f}_{\cdot,\alpha}(\lambda))(j-1) = \int_0^{\infty} \frac{s^{j-1}}{(j-1)!}e^{-(1+\lambda^{\alpha})s}\,ds = \frac{1}{(1+\lambda^{\alpha})^{j}}.$$

Define $\lambda := \dfrac{z-1}{(z^{\alpha}-(z-1)^{\alpha})^{1/\alpha}}$ where $|z|>1$. Hence

$$\mathcal{P}(\widehat{f}_{\cdot,\alpha}(\lambda))(j-1) = \left(\frac{1}{1+\dfrac{(z-1)^{\alpha}}{z^{\alpha}-(z-1)^{\alpha}}}\right)^{j} = \left(\frac{z^{\alpha}-(z-1)^{\alpha}}{z^{\alpha}}\right)^{j} = \widetilde{\ell}_{\alpha}(z,j).$$

□

Remark 3.5. Note that the non-negativity of $k^{\beta}(n)$ for $\beta > 0$ and $\ell_{\alpha}(n,j)$ for all $n \in \mathbb{N}_0$ and $j \in \mathbb{N}$ imply that the convolution $(k^{\beta} * \ell_{\alpha}(\cdot,j))(n)$ also is non-negative for all $n \in \mathbb{N}_0$ and $j \in \mathbb{N}$.

4. C-semigroups, (α,ν)-resolvent sequences and subordination

In this section, we introduce the notion of discrete C-semigroup and present some interesting properties. Moreover, we introduce the notion of (α,ν)-resolvent sequences and prove a subordination formula.

Let C be a bounded and injective operator defined on a complex Banach space X. Suppose that an strongly continuous operator-valued sequence $\{\mathcal{T}(n)\}_{n\in\mathbb{N}_0} \subset \mathcal{B}(X)$ satisfies the following conditions:

- (i) $\mathcal{T}(0) = C$,
- (ii) $C\mathcal{T}(n+m) = \mathcal{T}(n)\mathcal{T}(m)$ for $n,m \in \mathbb{N}$.

In analogy with the continuous case [8], we say that the family $\{\mathcal{T}(n)\}_{n\in\mathbb{N}_0}$ is a *discrete C-semigroup.*

From the definition, it is clear that $\mathcal{T}(n)$ commutes with C. Moreover, by simple iteration, we found that any discrete C-semigroup have the form

$$\mathcal{T}(n) = [C^{-1}\mathcal{T}(1)]^{n}C = C^{-(n-1)}\mathcal{T}(1)^{n}, \quad n \in \mathbb{N}_0. \tag{4.1}$$

From now on, A will denote a closed linear operator with domain $D(A)$ defined on a Banach space X and $\rho(A)$ will denote its resolvent set.

Proposition 4.1. Let $\{\mathcal{T}(n)\}_{n\in\mathbb{N}_0} \subset \mathcal{B}(X)$ be strongly continuous and satisfying the following properties

(*i*) $\mathcal{T}(n)x \in D(A)$ for all $x \in X$;

(*ii*) $A\mathcal{T}(n)x = \mathcal{T}(n)Ax$ for each $x \in D(A)$ and $n \in \mathbb{N}_0$;

(*iii*) $\mathcal{T}(n)x = x + A\displaystyle\sum_{j=0}^{n}\mathcal{T}(j)x$, for all $n \in \mathbb{N}_0$ and $x \in X$.

Then, $1 \in \rho(A)$ and $\{\mathcal{T}(n)\}_{n\in\mathbb{N}_0} \subset \mathcal{B}(X)$ is a discrete C-semigroup with $C := (I-A)^{-1}$.

Proof. With $n = 0$ the property (*iii*) gives

$$\mathcal{T}(0)x = x + A\mathcal{T}(0)x, \quad x \in X,$$

which, together with (*ii*), implies that $1 \in \rho(A)$ and $T(0) = (I-A)^{-1}$. Using the identity

$$(I-A)^{-1} - I = A(I-A)^{-1}, \tag{4.2}$$

we have that

$$\begin{aligned}\mathcal{T}(1)x =& x + A\mathcal{T}(0)x + A\mathcal{T}(1)x \quad n \in \mathbb{N}_0, \quad x \in X\\ =&\mathcal{T}(0)x + A\mathcal{T}(1)x,\end{aligned}$$

or

$$\mathcal{T}(1)x = (I-A)^{-2}x.$$

Iterating (*iii*) we find, more generally,

$$\mathcal{T}(n)x = (I-A)^{-(n+1)}x = (I-A)^{-n}C = [C^{-1}\mathcal{T}(1)]^nC \tag{4.3}$$

which proves that $\{\mathcal{T}(n)\}_{n\in\mathbb{N}_0} \subset \mathcal{B}(X)$ is a C-semigroup with $C := (I-A)^{-1}$.

□

Remark 4.2. We say that the family $\{\mathcal{T}(n)\}_{n\in\mathbb{N}_0} \subset \mathcal{B}(X)$ satisfying (*i*) − (*iii*) in the above Proposition is a discrete C-semigroup *generated by A.*

The following result shows an interesting new interpretation of one of the main results in the reference [22]. The striking point that shows the next theorem is that, in strong contrast with the continuous case, the natural family of operators behind of the well posedness of the discrete abstract Cauchy problem of first order is a discrete C-semigroup instead of a discrete semigroup, which was the first attempt in [22].

Theorem 4.3. Let $\{\mathcal{T}(n)\}_{n\in\mathbb{N}_0} \subset \mathcal{B}(X)$ be a discrete C-semigroup generated by A. Then the discrete-time abstract Cauchy problem of first order

$$\Delta u(n) = Au(n+1), \quad n \in \mathbb{N}_0,$$

with initial condition $u(0) = u_0 \in X$ admits the unique solution

$$u(n) = C^{-1}\mathcal{T}(n)u_0.$$

Proof. Note that $k^1(n) = 1$ for all $n \in \mathbb{N}_0$. By (2.8) and (2.2), we get

$$\begin{aligned}\Delta\mathcal{T}(n)x &= \Delta k^1(n)x + A\Delta\sum_{j=0}^{n}\mathcal{T}(j)x \\ &= A\left(\sum_{j=0}^{n+1}\mathcal{T}(j)x - \sum_{j=0}^{n}\mathcal{T}(j)x\right) \\ &= A\mathcal{T}(n+1)x.\end{aligned}$$

We define $u(n) := C^{-1}\mathcal{T}(n)u_0$. Since $\mathcal{T}(n)x \in D(A)$ for all $x \in X$ and $n \in \mathbb{N}_0$, we obtain $u(n) \in D(A)$ for all $n \in \mathbb{N}_0$. From the above identity, it is clear that $\Delta u(n) = Au(n+1)$. Finally, since $T(0) = C$, we obtain $u(0) = u_0$. ☐

Motivated by Proposition 4.1, we now introduce the following sequence of bounded and linear operators that is the discrete counterpart of the concept of resolvent families of operators for fractional evolution equations in continuous time. See [20] for a recent review of this concept and its main properties, and the reference [21] for their applications to nonlinear fractional evolution equations in the setting of Banach spaces.

Definition 4.4. Let $\alpha, \nu > 0$ be given. An operator-valued sequence $\{S_{\alpha,\nu}(n)\}_{n\in\mathbb{N}_0} \subset \mathcal{B}(X)$ is called a discrete (α,ν)-resolvent sequence generated by A if it satisfies the following conditions:

- *(i)* $S_{\alpha,\nu}(n)x \in D(A)$ for all $x \in X$ and $S_{\alpha,\nu}(n)Ax = AS_{\alpha,\nu}(n)x$ for each $n \in \mathbb{N}_0$ and $x \in D(A)$;
- *(ii)* $S_{\alpha,\nu}(n)x = k^{\nu}(n)x + A(k^{\alpha} * S_{\alpha,\nu})(n)x$ for all $n \in \mathbb{N}_0$ and each $x \in X$.

Note that $S_{1,1}(n) = \mathcal{T}(n)$ is the C-semigroup generated by A. The case $\alpha = \nu$ was introduced in [2, Definition 3.1] and used, among others, in the references [3, Section 2] and [17] in connection with linear and nonlinear fractional abstract difference equations. In particular, in [3, Proposition 2.2] it was proved that if A is a bounded operator with norm less than 1, then the following representation holds:

$$S_{\alpha,\alpha}(n) = \sum_{j=0}^{\infty} k^{\alpha(j+1)}(n)A^j.$$

The following is the main result of this paper and shows a striking relation between discrete (α,ν)-resolvent sequences and discrete C-semigroups. It allows a representation of a discrete (α,ν)-resolvent sequence in terms of a discrete C-semigroup generated by A. This result extend and improve [2, Theorem 3.2] and [3, Theorem 2.3].

Theorem 4.5. Let $0 < \alpha < 1$, $0 < \nu$ be given. Let $\{\mathcal{T}(n)\}_{n\in\mathbb{N}}$ be a discrete C-semigroup generated by A. Then the family

$$S_{\alpha,\nu}(n)x = \sum_{j=0}^{\infty}(k^{\nu-\alpha} * \ell_\alpha(\cdot, j))(n)\mathcal{T}(j)x, \quad n \in \mathbb{N}_0, \tag{4.4}$$

is a discrete (α, ν)-resolvent sequence generated by A.

Proof. From definition of C-semigroup and the fact that A is closed, we have that $S_{\alpha,\nu}(n)x \in D(A)$ for all $x \in X$ and $S_{\alpha,\nu}(n)Ax = AS_{\alpha,\nu}(n)x$ for each $x \in D(A)$. The group property of k^α shows that

$$A(k^\alpha * S_{\alpha,\nu})(n)x = \sum_{j=0}^{\infty}(k^\nu * \ell_\alpha(\cdot, j))(n)A\mathcal{T}(j)x.$$

Note that (4.3) and the identity $A(I-A)^{-1} = (I-A)^{-1}-I$ imply that $ACx = C^{j+1}x-C^jx$ for all $x \in X$. Therefore,

$$\begin{aligned} A(k^\alpha * S_{\alpha,\nu})(n)x &= \sum_{j=0}^{\infty}(k^\nu * \ell_\alpha(\cdot, j))(n)[C^{j+1} - C^j]x \\ &= \sum_{j=0}^{\infty}(k^\nu * \ell_\alpha(\cdot, j))(n)C^{j+1}x - \sum_{j=0}^{\infty}(k^\nu * \ell_\alpha(\cdot, j))(n)C^jx. \end{aligned}$$

Hence

$$\begin{aligned} A(k^\alpha * S_{\alpha,\nu})(n)x &= \sum_{j=0}^{\infty}(k^\nu * \ell_\alpha(\cdot, j))(n)C^{j+1}x - \sum_{j=1}^{\infty}(k^\nu * \ell_\alpha(\cdot, j))(n)C^jx - k^\nu(n)x \\ &= \sum_{j=0}^{\infty}(k^\nu * \ell_\alpha(\cdot, j))(n)C^{j+1}x - \sum_{j=0}^{\infty}(k^\nu * \ell_\alpha(\cdot, j+1))(n)C^{j+1}x - k^\nu(n)x \\ &= \sum_{j=0}^{\infty}(k^\nu * [\ell_\alpha(\cdot, j)(n) - \ell_\alpha(\cdot, j+1)])(n)C^{j+1}x - k^\nu(n)x, \end{aligned}$$

where in the first equality we have used (3.1) (for $j = 0$). Applying the Proposition 3.4, item (vi), we get

$$A(k^\alpha * S_{\alpha,\nu})(n)x = \sum_{j=0}^{\infty}(k^{\nu-\alpha} * \ell_\alpha(\cdot, j))(n)C^{j+1}x - k^\nu(n)x = S_{\alpha,\nu}(n)x - k^\nu(n)x.$$

It proves that $\{S_{\alpha,\nu}(n)\}_{n\in\mathbb{N}_0}$ is a discrete (α, ν)-resolvent sequence generated by A. □

The following consequence establishes an interesting link between the stability property of the C-semigroup with those of the (α, ν)-resolvent sequence, both generated by the same operator A.

Proposition 4.6. Let $0 < \alpha < 1$ and $\alpha \leq \nu$. Assume that A is the generator of a discrete C-semigroup $\{\mathcal{T}(n)\}_{n\in\mathbb{N}}$ and there exist constants $M, \omega > 0$ such that

$$\| \mathcal{T}(n) \| \leq M(1+\omega)^{-(n+1)}, \quad for\ all\ n \in \mathbb{N}_0.$$

Then the discrete (α,ν)-resolvent sequence $\{S_{\alpha,\nu}(n)\}_{n\in\mathbb{N}}$ generated by A satisfies

$$\| S_{\alpha,\nu}(n) \| \leq M\mathcal{E}_{\alpha,\nu}(-\omega, n).$$

In particular, if $0 < \nu < 1$, then $\| S_{\alpha,\nu}(n) \| \to 0$ as $n \to \infty$.

Proof. Let $\nu \geq \alpha$. Proposition 3.4 (iv) and $(viii)$ together with (4.4) and (2.5) can be used to obtain that

$$\begin{aligned}
\| S_{\alpha,\nu}(n) \| &= \| \sum_{j=0}^{\infty} (k^{\nu-\alpha} * \ell_\alpha(\cdot, j))(n)\mathcal{T}(j) \| \leq \sum_{j=0}^{\infty} \| (k^{\nu-\alpha} * \ell_\alpha(\cdot, j))(n)\mathcal{T}(j) \| \\
&= \sum_{j=0}^{\infty} (k^{\nu-\alpha} * \ell_\alpha(\cdot, j))(n) \| \mathcal{T}(j) \| \leq M \sum_{j=0}^{\infty} (k^{\nu-\alpha} * \ell_\alpha(\cdot, j))(n)\ (1+\omega)^{-(j+1)} \\
&= M(k^{\nu-\alpha} * \mathcal{E}_{\alpha,\alpha}(-\omega, \cdot))(n).
\end{aligned}$$

Now, the group property of k^α implies

$$\begin{aligned}
(k^{\nu-\alpha} * \mathcal{E}_{\alpha,\alpha}(-\omega), \cdot))(n) &= \sum_{h=0}^{n} k^{\nu-\alpha}(n-h) \sum_{j=0}^{\infty} (-\omega)^j k^{\alpha j+\alpha}(h) \\
&= \sum_{j=0}^{\infty} (-\omega)^j (k^{\nu-\alpha} * k^{\alpha j+\alpha})(n) \\
&= \sum_{j=0}^{\infty} (-\omega)^j k^{\alpha j+\nu}(n) \\
&= \mathcal{E}_{\alpha,\nu}(-\omega, n),
\end{aligned}$$

whence it follows that

$$\| S_{\alpha,\nu}(n) \| \leq M\mathcal{E}_{\alpha,\nu}(-\omega, n).$$

Hence, using (2.9), we can get the first desired conclusion. In order to prove the asymptotic behavior for $0 < \nu < 1$, we notice that from Theorem 2.8 we have

$$\mathcal{E}_{\alpha,\nu}(-\omega, n) = \int_0^\infty p_n(t) t^{\nu-1} E_{\alpha,\nu}(-\omega t^\alpha) dt. \tag{4.5}$$

Observe that from [26, Theorem 1.6] the following estimate

$$|E_{\alpha,\nu}(-\omega t^\alpha)| \leq \frac{C}{1+\omega t^\alpha}$$

holds. Using the above estimate in (4.5) we obtain

$$|\mathcal{E}_{\alpha,\nu}(-\omega,n)| \leq \int_0^\infty \frac{e^{-t}}{n!} t^{n+\nu-1} E_{\alpha,\nu}(-\omega t^\alpha) dt \leq \int_0^\infty \frac{e^{-t}}{n!} t^{n+\nu-1} \frac{C}{1+\omega t^\alpha} dt$$
$$\leq \frac{C}{n!\omega} \int_0^\infty \frac{e^{-t}}{n!} t^{n+\nu-\alpha-1} dt = \frac{C}{\omega} \frac{\Gamma(n+\nu-\alpha)}{n!},$$

where in the last equality we used the property (2.27). Now, taking into account that $\lim_{n\to\infty} \frac{\Gamma(n+\gamma)}{\Gamma(n)n^\gamma} = 1$ for all $\gamma \in \mathbb{C}$, and since $0 < \nu < 1$, we deduce for $\gamma := \nu - \alpha$ that

$$\frac{\Gamma(n+\nu-\alpha)}{n!} = \frac{\Gamma(n+\nu-\alpha)}{\Gamma(n)n^{\nu-\alpha}} \frac{1}{n^{1-(\nu-\alpha)}} \to 0 \quad (n \to \infty).$$

This proves the claim and proof is finished. □

5. The Hilfer fractional difference operator

Taking into account Definitions 2.2 and 2.4, and the definition introduced by R. Hilfer in [18], we define in this section the Hilfer fractional difference operator $\Delta^{\alpha,\beta}$ of order $0 < \alpha$ and type $0 \leq \beta \leq 1$ as follows.

Definition 5.1. The Hilfer fractional difference $\Delta^{\alpha,\beta}$ of order $\alpha > 0$ and type $0 \leq \beta \leq 1$ of a sequence $f \in s(\mathbb{N}_0; X)$ is defined by

$$\Delta^{\alpha,\beta} f(n) := \Delta^{-\beta(m-\alpha)} \left(\Delta^m \left(\Delta^{-(m-\alpha)(1-\beta)} f\right)\right)(n), \quad n \in \mathbb{N}_0,$$

where $m - 1 < \alpha \leq m$, $m := \lceil \alpha \rceil$.

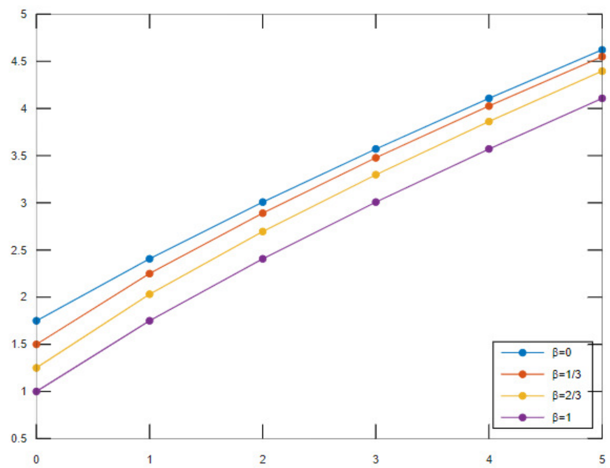

Figure 3. $\Delta^{\alpha,\beta} f(n)$ where $\alpha = 1/4$ and $f(n) = n+1$

.

Note that, as expected

$$\begin{aligned}\Delta^{m,\beta} f(n) &= \Delta^m f(n),\\ \Delta^{\alpha,0} f(n) &= {}_{RL}\Delta^\alpha f(n),\\ \Delta^{\alpha,1} f(n) &= {}_{C}\Delta^\alpha f(n).\end{aligned}$$

In other words, the two parameter family of operators $\Delta^{\alpha,\beta}$ of order $\alpha > 0$ and type $0 \le \beta \le 1$ allow us to interpolate between the Riemann-Liouville and the Caputo fractional difference operators.

We remark that Definition 5.1 can be compared with that recently introduced in [16, Definition 3.1], but first we note that in the definition given by the authors in [16] there is a minor imprecision that we could like to clarify. It concerns with the compatibility of the different operators used, because the meaning of the operator Δ used in [16, Definition 3.1] is not precise at all. The definition must read as follows (in case $0 < \mu < 1$, and in the general case is completely analogous).

$$\Delta_a^{\mu,\nu} = \Delta_{a+(1-\nu)(1-\mu)}^{-\nu(1-\mu)} \circ \Delta_{a+(1-\nu)(1-\mu)} \circ \Delta_a^{-(1-\nu)(1-\mu)}, \quad 0 \le \nu \le 1, \tag{5.1}$$

where

$$\Delta_a^{-\alpha} f(t) = \frac{1}{\Gamma(\alpha)} \sum_{s=a}^{t-\alpha} (t-s-1)^{(\alpha-1)} f(s),$$

with $t \in \mathbb{N}_{a+\alpha}$ and $t^{(\alpha)} := \frac{\Gamma(t+1)}{\Gamma(t-\alpha+1)}$, $\alpha > 0$. For the definition of the operator $\Delta_{a+(1-\nu)(1-\mu)}$ in (5.1) see (2.4). In order to compare both definitions, we will need the following lemma.

Lemma 5.2. For all $\alpha > 0$ and $b \in \mathbb{R}$, we have $\tau_{b+\alpha} \circ \Delta_b^{-\alpha} = \Delta^{-\alpha} \circ \tau_b$.

Proof. By Definition, for any $f \in s(\mathbb{N}_b, X)$ and all $n \in \mathbb{N}_0$ we have

$$\begin{aligned}\tau_{b+\alpha} \circ \Delta_b^{-\alpha} f(n) &= \Delta_b^{-\alpha} f(n+b+\alpha) = \frac{1}{\Gamma(\alpha)} \sum_{j=0}^{n} (\alpha+n-j-1)^{(\alpha-1)} f(b+j)\\ &= \sum_{j=0}^{n} \frac{\Gamma(\alpha+n-j)}{\Gamma(\alpha)\Gamma(n-j+1)} f(b+j) = \sum_{j=0}^{n} k^\alpha(n-j) f(b+j) = \Delta^{-\alpha} \circ \tau_b f(n).\end{aligned}$$

□

Lemma 5.2 shows that the following diagram is commutative:

$$\begin{array}{ccc} s(\mathbb{N}_b;X) & \xrightarrow{\Delta_b^{-\alpha}} & s(\mathbb{N}_{b+\alpha};X)\\ \downarrow \tau_b & & \downarrow \tau_{b+\alpha}\\ s(\mathbb{N}_0;X) & \xrightarrow{\Delta^{-\alpha}} & s(\mathbb{N}_0;X). \end{array} \tag{5.2}$$

With this preliminaries, we prove the following transference principle that generalizes Theorem 4.1 in [11].

Theorem 5.3. For any $0 < \mu < 1$, $0 \leq \nu \leq 1$ and $a \in \mathbb{R}$ we have

$$\Delta^{\mu,\nu} = \tau_{a+(1-\mu)(1-\nu)} \circ \Delta_a^{\mu,\nu} \circ \tau_{-a}.$$

Proof. Using (2.3) with $m = 1$ we have

$$\Delta_a^{\mu,\nu} = \Delta_{a+(1-\nu)(1-\mu)}^{-\nu(1-\mu)} \circ \tau_{-(a+(1-\nu)(1-\mu))} \circ \Delta \circ \tau_{a+(1-\nu)(1-\mu)} \circ \Delta_a^{-(1-\nu)(1-\mu)}. \tag{5.3}$$

We now employ (5.2) first with $b = a + (1-\mu)(1-\nu)$, $\alpha = \nu(1-\mu)$ and then with $b = a$, $\alpha = (1-\nu)(1-\mu)$ to obtain

$$\Delta_{a+(1-\nu)(1-\mu)}^{-\nu(1-\mu)} = \tau_{-a-(1-\mu)(1-\nu)} \circ \Delta^{-\nu(1-\mu)} \circ \tau_{a+(1-\mu)(1-\nu)} \tag{5.4}$$

and

$$\Delta_a^{-(1-\nu)(1-\mu)} = \tau_{-a-(1-\nu)(1-\mu)} \circ \Delta^{-(1-\nu)(1-\mu)} \circ \tau_a. \tag{5.5}$$

Replacing (5.4) and (5.5) in (5.3), we obtain

$$\begin{aligned} \Delta_a^{\mu,\nu} &= \tau_{-a-(1-\mu)(1-\nu)} \circ \Delta^{-\nu(1-\mu)} \circ \tau_{a+(1-\mu)(1-\nu)} \circ \tau_{-(a+(1-\nu)(1-\mu))} \circ \Delta \\ &\quad \circ \tau_{a+(1-\nu)(1-\mu)} \circ \tau_{-a-(1-\nu)(1-\mu)} \circ \Delta^{-(1-\nu)(1-\mu)} \circ \tau_a \\ &= \tau_{-a-(1-\mu)(1-\nu)} \circ \Delta^{-\nu(1-\mu)} \circ \Delta \circ \Delta^{-(1-\nu)(1-\mu)} \circ \tau_a = \tau_{-a-(1-\mu)(1-\nu)} \circ \Delta^{\mu,\nu} \circ \tau_a, \end{aligned}$$

proving the theorem. □

Motivated by Theorem 2.5, we obtain the following relation between the Hilfer and Riemann-Liouville fractional difference operators.

Theorem 5.4. Let $n \in \mathbb{N}_0$ and $0 \leq \beta \leq 1$. For each $\alpha > 0$ and $f \in s(\mathbb{N}_0; X)$, we have

$$\Delta^{\alpha,\beta} f(n) = {}_{RL}\Delta^{\alpha} f(n) - \sum_{i=0}^{m-1} (-1)^i \binom{m}{i} \sum_{j=0}^{m-1-i} k^{\beta(m-\alpha)}(n+m-j-i) \Delta^{-(m-\alpha)(1-\beta)} f(j),$$

where $m = \lceil \alpha \rceil$.

Proof. Define $\omega(n) := \Delta^{-(1-\alpha)(1-\beta)}f(n)$. Then, using the definitions, we obtain the following identities

$$\begin{aligned}
\Delta^{\alpha,\beta}f(n) &= \sum_{j=0}^{n} k^{\beta(m-\alpha)}(n-j)\Delta^m\omega(j)\\
&= \sum_{j=0}^{n} k^{\beta(m-\alpha)}(n-j)\sum_{i=0}^{m}(-1)^i\binom{m}{i}\omega(j+m-i)\\
&= \sum_{i=0}^{m}(-1)^i\binom{m}{i}\sum_{j=0}^{n} k^{\beta(m-\alpha)}(n-j)\omega(j+m-i)\\
&= \sum_{i=0}^{m}(-1)^i\binom{m}{i}\sum_{j=m-i}^{n+m-i} k^{\beta(m-\alpha)}(n+m-j-i)\omega(j)\\
&= \sum_{i=0}^{m}(-1)^i\binom{m}{i}\sum_{j=0}^{n+m-i} k^{\beta(m-\alpha)}(n+m-i-j)\omega(j)\\
&\qquad -\sum_{i=0}^{m-1}(-1)^i\binom{m}{i}\sum_{j=0}^{m-1-i} k^{\beta(m-\alpha)}(n+m-j-i)\omega(j)\\
&= \Delta^m(\Delta^{-\beta(m-\alpha)}\omega)(n)\\
&\qquad -\sum_{i=0}^{m-1}(-1)^i\binom{m}{i}\sum_{j=0}^{m-1-i} k^{\beta(m-\alpha)}(n+m-j-i)\omega(j).
\end{aligned}$$

Here, we have adopted the convention $\sum_{j=0}^{-1} f(j) = 0$. The conclusion follows from the property (2.12). ☐

It is instructive to look at the cases $0 < \alpha < 1$ and $1 < \alpha < 2$ for further developments.

Corollary 5.5. Let $0 \le \beta \le 1$. For $0 < \alpha < 1$, we have

$$\Delta^{\alpha,\beta}f(n) = {}_{RL}\Delta^{\alpha}f(n) - k^{\beta(1-\alpha)}(n+1)f(0), \qquad n \in \mathbb{N}_0, \tag{5.6}$$

and in case $1 < \alpha < 2$, we obtain

$$\Delta^{\alpha,\beta}f(n) = {}_{RL}\Delta^{\alpha}f(n) - k^{\beta(2-\alpha)}(n+1)\left[\Delta^{-(2-\alpha)(1-\beta)}f(1) - 2f(0)\right] + k^{\beta(2-\alpha)}(n+2)f(0),$$

for all $n \in \mathbb{N}_0$.

Concerning to the $\mathcal{Z}$-transform of the Hilfer fractional difference operator, we prove the following property.

Proposition 5.6. For $0 < \alpha < 1$ and $0 \le \beta \le 1$, let $y(n) := \Delta^{\alpha,\beta}f(n)$ where $f \in s(\mathbb{N}_0; X)$. Then

$$\widetilde{y}(z) = z^{1-\alpha}(z-1)^{\alpha}\tilde{f}(z) - z^{\beta(1-\alpha)+1}(z-1)^{\alpha-\nu}f(0), \quad |z| > 1,$$

where $\nu = \alpha + \beta(1-\alpha)$.

Proof. By Definition 5.1, we have

$$y(n) = \Delta^{-\beta(1-\alpha)}{}_{RL}\Delta^{\nu} f(n).$$

Let $r(n) := {}_{RL}\Delta^{\nu} f(n)$. From (2.15) we obtain

$$\widetilde{\Delta^{-\beta(1-\alpha)} r}(z) = z^{\beta(1-\alpha)}(z-1)^{\beta(\alpha-1)}\tilde{r}(z), \tag{5.7}$$

and, by (2.16),

$$\widetilde{r}(z) = z^{1-\nu}(z-1)^{\nu}\tilde{f}(z) - zf(0). \tag{5.8}$$

Substituting (5.8) in (5.7), we get the conclusion. □

Remark 5.7. The above proposition coincides with [16, Theorem 3.5] (and also rectifies the writing of the result in [16]), after using the transference principle given by Theorem 5.3. We will also need the following lemma that connects the Delta Laplace transform $\mathcal{L}_b$ [12, Definition 2.1] with the usual $\mathcal{Z}$-transform:

Lemma 5.8. Assume $f : \mathbb{N}_b \to X$. Then

$$\mathcal{L}_b\{f\}(z-1) = \frac{1}{z}\widetilde{(\tau_b f)}(z),$$

for all $z \in \mathbb{C} \setminus \{0\}$ such that the series defined by the $\mathcal{Z}$-transform converges.

Proof. We use [12, Theorem 2.2] and obtain

$$\mathcal{L}_b\{f\}(z-1) = \sum_{k=0}^{\infty} \frac{f(b+k)}{z^{k+1}} = \frac{1}{z}\sum_{k=0}^{\infty} \frac{\tau_b f(k)}{z^k} = \frac{1}{z}\widetilde{(\tau_b f)}(z).$$

□

Now, using Lemma 5.8, Theorem 5.3 and Proposition 5.6, we have the following identities

$$\begin{aligned}\mathcal{L}_{a+(1-\mu)(1-\nu)}\{\Delta_a^{\mu,\nu} f\}(z-1) &= \frac{1}{z}(\tau_{a+(1-\mu)(1-\nu)} \circ \Delta_a^{\mu,\nu} f\tilde{)}(z) = \frac{1}{z}(\Delta^{\mu,\nu} \circ \tau_a f\tilde{)}(z)\\ &= z^{1-\mu}(z-1)^{\mu}\frac{1}{z}\widetilde{(\tau_a f)}(z) - z^{\nu(1-\mu)}(z-1)^{-\nu(1-\mu)}(\tau_a f)(0)\\ &= z^{1-\mu}(z-1)^{\mu}\mathcal{L}_a\{f\}(z-1) - z^{\nu(1-\mu)}(z-1)^{-\nu(1-\mu)} f(a).\end{aligned}$$

Replacing $s = z-1$ we improve [16, Theorem 3.5] as follows:

$$\mathcal{L}_{a+(1-\mu)(1-\nu)}\{\Delta_a^{\mu,\nu} f\}(s) = (s+1)^{1-\mu}s^{\mu}\mathcal{L}_a\{f\}(s) - (s+1)^{\nu(1-\mu)}s^{-\nu(1-\mu)} f(a).$$

Now, using the Poisson transform, we can establish an important relation between the Hilfer fractional difference operator and the Hilfer fractional continuous operator. This result extends [22, Theorem 3.5].

Theorem 5.9. Let $u : [0,\infty) \to X$ be an absolutely integrable and bounded function. Then

$$\mathcal{P}({}_H D^{\alpha,\beta} u)(n+1) = \Delta^{\alpha,\beta}\mathcal{P}(u)(n), \quad n \in \mathbb{N}_0, \tag{5.9}$$

that is

$$\int_0^\infty p_{n+1}(t)_H D_t^{\alpha,\beta} u(t)\ dt = \Delta^{\alpha,\beta} u(n), \qquad n \in \mathbb{N}_0,$$

where $u(n) := \displaystyle\int_0^\infty p_n(t)u(t)\ dt$.

Proof. Taking the definition of continuous Hilfer derivative [18], multiplying by $p_n(t)$ and then integrating over $\mathbb{R}_+$, we obtain

$$\int_0^\infty p_{n+1}(t)_H D_t^{\alpha} u(t)\ dt = \int_0^\infty p_{n+1}(t)(g_{\beta(1-\alpha)} * {}_{RL}D_t^{\nu} u)(t)\ dt,$$

where $\nu = \alpha + \beta(1-\alpha)$. Theorem 3.4 of [22] shows that

$$\int_0^\infty p_{n+1}(t)(g_{\beta(1-\alpha)} * {}_{RL}D_t^{\nu} u)(t)\ dt = \sum_{j=0}^{n+1} a(n+1-j)S(j),$$

where

$$a(n) := \int_0^\infty p_n(t) g_{\beta(1-\alpha)}(t) dt \quad \text{and} \quad S(n) := \int_0^\infty p_n(t)_{RL} D_t^{\nu} u(t)\ dt.$$

Now, observe that the definition of k^γ in (2.6), shows that

$$a(n) = \frac{1}{n!\Gamma(\beta(1-\alpha))}\int_0^\infty e^{-t} t^{n+\beta(1-\alpha)-1}\ dt = \frac{\Gamma(n+\beta(1-\alpha))}{\Gamma(\beta(1-\alpha))n!} = k^{\beta(1-\alpha)}(n).$$

Therefore, [22, Theorem 3.5] gives

$$\begin{aligned}\int_0^\infty p_{n+1}(t)_H D_t^{\alpha} u(t)\ dt &= \sum_{j=0}^{n+1} k^{\beta(1-\alpha)}(n+1-j)S(j)\\ &= \Delta^{-\beta(1-\alpha)} S(n+1)\\ &= \Delta^{-\beta(1-\alpha)}{}_{RL}\Delta^{\nu} u(n).\end{aligned}$$

Hence the conclusion follows. □

We finish this section with the following result that shows the connection of resolvent sequences with solutions of fractional difference equations employing the notion of Hilfer fractional difference operator defined previously.

Theorem 5.10. Let $0 < \alpha < 1$, $0 \le \beta \le 1$ and $\nu = \alpha + \beta(1-\alpha)$. Suppose that A is the generator of an (α,ν)-resolvent sequence $\{S_{\alpha,\nu}(n)\}_{n\in\mathbb{N}_0}$ and $1 \in \rho(A)$. Then the fractional difference equation

$$\text{(5.10)} \qquad \begin{aligned}\Delta^{\alpha,\beta} u(n) &= A\left[u(n+1) - k^{\beta(1-\alpha)}(n+1)u_0\right], \quad n \in \mathbb{N}_0,\\ u(0) &= u_0 \in D(A),\end{aligned}$$

admits the unique solution

$$u(n) = S_{\alpha,\nu}(n)C^{-1}u_0, \quad n \in \mathbb{N}_0,$$

where $C = (I-A)^{-1}$.

Proof. Let $u(n) := S_{\alpha,\nu}(n)C^{-1}u_0$ for all $n \in \mathbb{N}_0$. By Definition 4.4, we have $u(n) \in D(A)$ and the following identities

$$\begin{aligned}(k^{1-\alpha} * S_{\alpha,\nu})(n)x &= k^{1+\beta(1-\alpha)}(n)x + A(k^1 * S_{\alpha,\nu})(n)x \\ &= k^{1+\beta(1-\alpha)}(n)x + A\sum_{j=0}^{n} S_{\alpha,\nu}(j)x, \quad n \in \mathbb{N}_0\end{aligned}$$

holds. Applying the operator Δ to both sides of the last identity and (2.14), we get

$$_{RL}\Delta S_{\alpha,\nu}(n)x = k^{\beta(1-\alpha)}(n+1)x + AS_{\alpha,\nu}(n+1)x, \quad n \in \mathbb{N}_0.$$

Then, by Corollary 5.5 and the identity $C - I = AC$, we obtain

$$\begin{aligned}\Delta^{\alpha,\beta} S_{\alpha,\nu}(n) &= k^{\beta(1-\alpha)}(n+1)x + AS_{\alpha,\nu}(n+1)x - k^{\beta(1-\alpha)}(n+1)S_{\alpha,\nu}(0)x \\ &= AS_{\alpha,\nu}(n+1)x - k^{\beta(1-\alpha)}(n+1)(C-I)x \\ &= A\left[S_{\alpha,\nu}(n+1)x - k^{\beta(1-\alpha)}(n+1)Cx\right], \quad n \in \mathbb{N}_0.\end{aligned}$$

Hence, u solves (5.10), which proves the theorem. □

Remark 5.11. Under the same hypothesis of the previous theorem, note that $u(n) := S_{\alpha,\alpha}(n)C^{-1}u_0$ solves

$$\begin{aligned}{}_{RL}\Delta^{\alpha} u(n) &= Au(n+1), \quad n \in \mathbb{N}_0, \\ u(0) &= u_0 \in D(A),\end{aligned}$$

and $u(n) := S_{\alpha,1}(n)C^{-1}u_0$ solves

$$\begin{aligned}{}_{C}\Delta^{\alpha} u(n) &= A\left[u(n+1) - k^{1-\alpha}(n+1)u_0\right], \quad n \in \mathbb{N}_0, \\ u(0) &= u_0 \in D(A).\end{aligned}$$

The following corollary is an interesting but direct consequence of the theory developed until now. It takes in consideration the subordination formula stated previously.

Corollary 5.12. Let $0 < \alpha < 1$ and $0 < \beta < 1$ be given. Assume that A generates a discrete C-semigroup $\{\mathcal{T}(n)\}_{n\in\mathbb{N}_0}$ such that $\|\mathcal{T}(n)\| \le M(1+\omega)^{-n}$ for some $M, \omega > 0$. Then the fractional difference equation

$$(5.11) \qquad \begin{aligned}\Delta^{\alpha,\beta} u(n) &= A\left[u(n+1) - k^{\beta(1-\alpha)}(n+1)u_0\right], \quad n \in \mathbb{N}_0, \\ u(0) &= u_0 \in D(A),\end{aligned}$$

admits the unique solution

$$u(n) = \sum_{j=0}^{\infty} (k^{\beta(1-\alpha)} * \ell_\alpha(\cdot, j))(n)\mathcal{T}(j)C^{-1}u_0, \quad n \in \mathbb{N}_0.$$

Moreover, $u(n) \to 0$ as $n \to \infty$.

Proof. Since A generates a discrete C-semigroup we have $1 \in \rho(A)$. Let $\nu := \alpha + \beta(1-\alpha)$. By Theorem 4.5 (subordination) we have that A generates a discrete (α,ν)-resolvent family $\{S_{\alpha,\nu}(n)\}_{n\in\mathbb{N}_0}$ given by

$$S_{\alpha,\nu}(n)x = \sum_{j=0}^{\infty}(k^{\nu-\alpha} * \ell_\alpha(\cdot,j))(n)\mathcal{T}(j)x, \quad n \in \mathbb{N}_0. \tag{5.12}$$

From Theorem 5.10 the unique solution is then given by

$$u(n) = S_{\alpha,\nu}(n)C^{-1}u_0, \quad n \in \mathbb{N}_0,$$

where $C = (I-A)^{-1}$. Since $\nu \geq \alpha$, Proposition 4.6 implies that $\|S_{\alpha,\nu}(n)\| \leq M\mathcal{E}_{\alpha,\nu}(-\omega,n)$. Note that for $0<\beta<1$ we have $\alpha(1-\beta) < 1-\beta$ and hence $0<\nu<1$. It follows from Proposition 4.6 that $u(n) \to 0$ as $n \to \infty$, finishing the proof. □

We end this paper with the following concrete example.

Example 5.13. Let us consider the discrete fractional diffusion problem

$$\begin{aligned} {}_{RL}\Delta^\alpha v(n,x) &= \Delta_x v(n+1,x),\ n \in \mathbb{N}_0,\ x\in\mathbb{R}, \\ u(0,x) &= f(x), \end{aligned} \tag{5.13}$$

where $0<\alpha<1$, Δ_x is the classical one dimensional Laplacian operator on $L^p(\mathbb{R})$ with maximal domain defined by $\Delta_x = \frac{\partial^2}{\partial x^2}$, u is a function defined on $\mathbb{N}\times\mathbb{R}$ and f is defined on $\mathbb{R}$. Recently, Abadias and Alvarez proved in [1] that the problem corresponding to $\alpha = 1$, namely

$$\begin{aligned} u(n+1,x) - u(n,x) &= \Delta_x u(n+1,x), \qquad n\in\mathbb{N}_0,\ x\in\mathbb{R}, \\ u(0,x) &= f(x), \end{aligned}$$

has the unique explicit solution

$$u(n,x) = \mathcal{T}(n)C^{-1}f(x) := (\mathcal{G}_{n+1} * f)(x), \quad n\in\mathbb{N}_0,\ x\in\mathbb{R},$$

where $C^{-1} = I - \Delta_x$ and

$$\mathcal{G}_{n+1}(x) = \frac{2}{\Gamma(n+1)\sqrt{4\pi}}\left(\frac{|x|}{2}\right)^{n+1/2}\mathcal{K}_{n+1/2}(|x|), \qquad n\in\mathbb{N}_0, \quad x\in\mathbb{R}.$$

Here, $\mathcal{K}_m$ denotes the Bessel functions of imaginary argument (also called MacDonald's functions or modified cylinder functions). Then, by Theorem 4.5 we have that Δ_x generates an (α,α)-resolvent sequence $\{S_{\alpha,\alpha}(n)\}_{n\in\mathbb{N}_0}$ and moreover, by Remark 5.11, the solution of (5.13) is given by

$$v(n,x) = S_{\alpha,\alpha}(n)C^{-1}f(x) = S_{\alpha,\alpha}(n)f(x) - S_{\alpha,\alpha}(n)f'(x) = \sum_{j=0}^{\infty}\ell_\alpha(n,j)(\mathcal{G}_n * (f-f'))(x),$$

whenever $f'(x)$ exists.

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